Models, Algorithms and Data (MAD): Introduction to Computing

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Set 9

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In this exercise, we cover the basics of probability theory and we will apply Monte Carlo integration. It follows a reminder of the fundamental definitions from probability theory that were given in the lecture:

Probability Density Function: For a continuous random variable X with range $[a, b]$, the probability density function $p(x')$, is defined such that the probability P of the random variable X to take a value smaller than x is given by

$$P(X \leq x) = \int_{-\infty}^{x} p(x')\mathrm{d}x' \tag{1}$$

Usually this is denoted by $F_X(x) = P(X \leq x)$ and called **cumulative distribution function**.

Expectation Value: For a continuous random variable X with range $[a, b]$ and probability density function $p(x)$, the expected value of $h(X)$ is defined as

$$\mathbb{E}[h(X)] = \int_a^b h(x)p(x)dx \tag{2}$$

A special case is $h(X) = X$, which gives the mean $\mu = \mathbb{E}[X]$.

Variance: For a continuous random variable X the variance of X is:

$$\mathrm{Var}[X] = \mathbb{E}[(X - E[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{3}$$

We realize that this is an expectation value with $h(X) = (X - \mu)^2$, i.e. the variance is the expected mean squared distance from the mean μ.

Question 1: Random Variables: Probabilities, Expectation, Variance

You arrive in a building and are about to take an elevator to your floor. Once you call the elevator, it will take between 0 and 40 seconds to arrive. The duration until the elevator arrives can be seen as a random variable T. For the following we assume the time of arrival $t \in \mathbb{R}$ to be uniformly distributed between 0 and 40 seconds, i.e. $t \sim \mathcal{U}([0, 40])$.

a) Write down the probability density function $p(t)$ for the random variable T.

b) Calculate the probability $P(T \leq t)$ that elevator takes up to 15 seconds to arrive.

c) Calculate the mean value $\mathbb{E}[T]$ that the elevator takes to arrive.

d) Calculate the variance $\mathrm{Var}[T]$ the elevator takes to arrive.

Question 2: Random Variables: Expectation and Variance

Let X be a random variable that follows the exponential distribution $X \sim \mathrm{Exp}(\lambda)$. The range of X is $[0, \infty)$ and its `pdf` is $p(x) = \lambda \mathrm{e}^{-\lambda x}$.

a) For any positive integer n, prove that:

$$E[X^n] = \frac{n}{\lambda} E[X^{n-1}]$$

b) Compute the expectation value $\mathbb{E}[X]$ of X.

c) Compute the variance $\mathrm{Var}[X]$ of X.

Question 3: Monte Carlo Integration

a) Write a pseudocode to calculate the overlapping area of the two circles shown in Fig. 1 using Monte Carlo Integration. Assume that you have access to a function `random()` which returns a uniformly distributed random number in the interval $[0, 1]$ and make use of it.

b) What would add in your pseudocode if you wanted to estimate the error of the Monte Carlo Integration? Answer qualitatively, do not write any pseudocode.

c) How does the error of the method change if you use 10 times more samples?

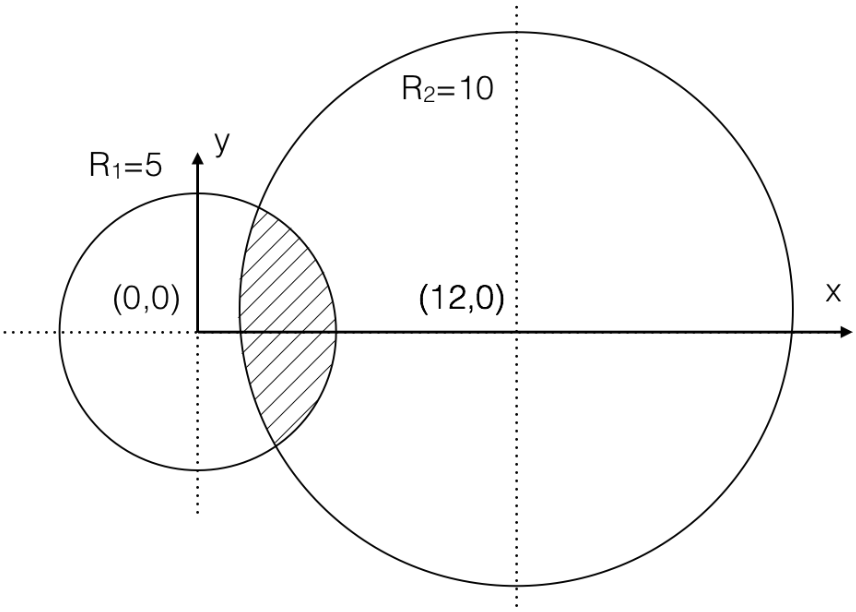

Figure 1: Sketch of two overlapping circles