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Solution Set 9

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Question 1: Random Variables: Probabilities, Expectation, Variance

a) We consider the probability distribution of the time of the arrival of the elevator to be uniform in the range $[a, b] = [0, 40]$ s. Therefore, the probability density of the random variable t is written as:

$$p(t) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq t \leq b \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

or, substituting a, b

$$p(t) = \begin{cases} \frac{1}{40}, & \text{if } 0 \leq t \leq 40 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

b) For any continuous PDF, we can calculate probabilities using integration. Specifically for a uniform distribution the probability of RV t being between $[c, d]$ is:

$$\mathrm{P}(c \leq t \leq d) = \int_c^d p(t)dt = \int_c^d \frac{1}{b-a}dt = \frac{d-c}{b-a} \tag{3}$$

In our specific case, the probability of time of arrival to be less than 15 s is given as:

$$\mathrm{P}(0 \leq t \leq 15) = \frac{15-0}{40-0} = \frac{15}{40} = \frac{3}{8} = 0.375 \tag{4}$$

c) The expectation value of the above uniform distribution is:

$$\mathrm{E[T]} = \int_a^b tp(t)dt = \int_a^b \frac{t}{b-a}dt = \frac{b-a}{2} = \frac{40-0}{2} = 20\,\text{seconds} \tag{5}$$

d) The variance of the above uniform distribution is:

$$\mathrm{Var[T]} = \mathrm{E[T^2]} - \mathrm{E^2[T]} = \int_a^b \frac{t^2}{b-a}dt - \left(\frac{b-a}{2}\right)^2 = \frac{(b-a)^2}{12} = \frac{(40-0)^2}{12} = \frac{400}{3} \tag{6}$$

Question 2: Random variables: Expectation and variance

a) For a random variable X^n with `PDF` $p(x) = \lambda e^{-\lambda x}$ for $x \geq 0$:

$$E[X^n] = \int_0^\infty x^n \cdot \lambda e^{-\lambda x} dx$$

Applying integral by parts with $u = x^n$, $dv = \lambda e^{-\lambda x}$, and $du = nx^{n-1}dx$, $v = -e^{-\lambda x}$, we obtain:

$$\begin{aligned}
E[X^n] &= \int_0^\infty x^n \cdot \lambda e^{-\lambda x} dx && (7)\\
&= [x^n \cdot (-e^{-\lambda x})]_0^\infty - \int_0^\infty (-e^{-\lambda x}) \cdot nx^{n-1} dx && (8)\\
&= 0 + n \int_0^\infty e^{-\lambda x} x^{n-1} dx && (9)\\
&= \frac{n}{\lambda} E[X^{n-1}]. && (10)
\end{aligned}$$

b) The expectation $E[X]$ is obtained from the formula we proved in question a, by substituting $n = 1$:

$$E[X] = \frac{1}{\lambda} E[X^0] = \frac{1}{\lambda} E[1] = \frac{1}{\lambda}$$

c) It is easier to compute the variance using the formula:

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2$$

From part (b) we found $E[X] = \frac{1}{\lambda}$. We compute $E[X^2]$ from the formula in part (a), substituting $n = 2$:

$$E[X^2] = \frac{2}{\lambda} E[X^1] = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

Thus, the variance is computed:

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Question 3: Monte Carlo Sampling Pseudocode

a) We choose to sample for our integration from an orthogonal domain, containing both circles. Therefore, with $(x_1,\, y_1)$ = (0,0) and R_1=5 and $(x_2,\, y_2)$ = (12,0) and R_2=10:

- x should be in the interval [-5,22]: We have to transform the random values taken by `random()` to this interval. Sampling in the x-direction will be:
  $x = \texttt{random()} * 27 - 5$
- y should be in the interval [-10,10]: We transform sampling in the y-direction:
  $y = \texttt{random()} * 20 - 10$

Algorithm 1 Monte Carlo integration for overlapping area

Input:

circle center x_2
circle radii R_1, R_2
number of Monte Carlo samples M

Output:

overlap area of 2 circles

Steps:

$R_1 = 5$
$R_2 = 10$
$x_2 = 12$
$M = 10000$

function OVERLAPMC(x_2, R_1, R_2, M)
  double area_rectangle= $(R_1 + x_2 + R_2) * 2. * R_2$
  int pts_inside $= 0$
  for int i:=0; i < M; i++ **do**
    double $x_i = (x_2 + R_2 + R_1) * \texttt{random()} - R_1$
    double $y_i = 2 * R_2 * \texttt{random()} - R_2$
    if $(x_i * x_i + y_i * y_i \leq R_1 * R_1)$ and $((x_i - x_2) * (x_i - x_2) + y_i * y_i \leq R_2 * R_2)$ **then**
      pts_inside ++
    end if
  end for
  return (double) pts_inside / M * area_rectangle
end function

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b) In general, if you want to compute the MC error, you have to keep both function evaluations AND function evaluations squared. In the specific case, the function is the unit function, if the random point is inside the overlapping area.

c) We know that MC error behaves as: $\epsilon_{MC} \sim \sqrt{\frac{|Var|}{M}}$. Given $M_2 = 10 \cdot M_1$, we have $\epsilon_{MC-2} = \frac{\epsilon_{MC-1}}{\sqrt{10}}$