Models, Algorithms and Data (MAD): Introduction to Computing

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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Prof. Dr. Jens Honore Walther
Dr. Georgios Arampatzis
ETH Zentrum, CLT
CH-8092 Zürich

Set 8

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Question 1: Gauss Quadrature

In this task, you will experience the advantages and limitations of the Gauss Quadrature. Gauss Quadrature uses a set of carefully designed integration points that can achieve high accuracy for numerical integration of sufficiently smooth functions. Here you will evaluate two integrals with three types of numerical integration methods: Trapezoidal rule, Newton-Cotes formula of order 2 and Gauss Quadrature of order 3 (see Table 1). All methods require three function evaluations. Perform numerical quadrature with:

(i) the composite Trapezoidal rule with uniform grid spacing and $n = 3$ points,

(ii) the Newton-Cotes (closed) formula for $n = 2$,

(iii) the Gauss Quadrature for $n = 3$,

for the following two integrals:

(a) $I = \int_1^3 x^6 - x^2 \sin(2x)\,\mathrm{d}x = 317.3442467$,

(b) $I = \int_0^2 1 - |x-1|\,\mathrm{d}x = 1$.

Which method does better for which integrand?

It follows a procedure of applying the Gauss Quadrature to evaluate $I = \int_a^b f(x)\,\mathrm{d}x$ on an arbitrary interval [a,b]:

1. Change the boundary of the integral from $[-1, 1]$ to $[a, b]$ using change of variables.
2. Read out the integration points z_i for z and the corresponding weights w_i from the table (see Table 1).
3. Evaluate the integral $I \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2} z_i + \frac{b+a}{2}\right)$, where n is the order of the Gauss Quadrature.

z_i	w_i
$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
0.0	$\frac{8}{9}$
$+\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

Table 1: Order three ($n = 3$) Gauss Quadrature integration points and weights.

Question 2: Adaptive quadrature

Apply the adaptive Trapezoid rule until we meet the specified relative tolerance. The error is estimated as $\epsilon(h/2) = G(h/2) - G(h)$ and the stopping criterion is

$$\epsilon(h/2) < tol \cdot \frac{h}{h_0},$$

where h_0 is the size if the initial interval $(b_0 - a_0)$.

Find the approximation of the integrals and calculate the error with respect to the analytical solution for both functions below.

a) $f(x) = x^2$, $a_0 = 0$, $b_0 = 1$ up to $tol = 0.15$

b) $f(x) = \cos(x)$, $a_0 = 0$, $b_0 = \pi/2$ up to $tol = 0.03$