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Solution Set 8

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Question 1: Gauss Quadrature

a) We first recall the integration rules for approximating $I = \int_a^b f(x)\,dx$ (we use $m = (a+b)/2$):

- Trapezoidal rule with 2 uniformly spaced intervals:

$$I \approx \frac{m-a}{2}[f(a)+f(m)] + \frac{b-m}{2}[f(m)+f(b)].$$

- Newton-Cotes with $n = 2$:

$$I \approx \frac{b-a}{6}\left[f(a)+4f(m)+f(b)\right].$$

We evaluate the 2 integrals using the 3 integration rules. Note the notation $0.\dot{5} = 0.5555555\ldots$:

(a) $I = \int_1^3 f(x)\,\mathrm{d}x$, where $f(x) = x^6 - x^2\sin(2x)$:

(i) Trapezoidal rule for $n = 3$:

$$\begin{aligned} I &\approx \frac{(1)}{2}[f(1)+2f(2)+f(3)] \\ &= 432.8299310 \end{aligned}$$

(ii) Newton-Cotes (closed) formula for $n = 2$:

$$\begin{aligned} I &\approx \frac{(2)}{6}[f(1)+4f(2)+f(3)] \\ &= 333.2380940 \end{aligned}$$

(iii) Gauss Quadrature for $n = 3$:
We first perform the change of variable:

$$I = \int_1^3 f(x)dx = \frac{3-1}{2}\int_{-1}^1 f\left(\frac{3-1}{2}(z-1)+3\right)dz = \int_{-1}^1 f\left(z+2\right)dz.$$

We now use the points and weights from the table:

$$\begin{aligned} I &\approx \sum_{i=1}^{3} w_i f(z_i+2) \\ &= 0.\dot{5}\cdot f(-0.7745966692+2) + 0.\dot{8}\cdot f(0.0+2) + 0.\dot{5}\cdot f(0.7745966692+2) \\ &= 317.2641516 \end{aligned}$$

(b) $I = \int_0^2 f(x)\,\mathrm{d}x$, where

$$f(x) = 1 - |x-1|$$

(i) Trapezoidal rule for $n = 3$:

$$\begin{aligned} I &\approx \frac{(1)}{2}[f(0) + 2f(1) + f(2)] \\ &= 1 \end{aligned}$$

(ii) Newton-Cotes (closed) formula for $n = 2$:

$$\begin{aligned} I &\approx \frac{(2)}{6}[f(0) + 4f(1) + f(2)] \\ &= 4/3 = 1.3333333\ldots \end{aligned}$$

(iii) Gauss Quadrature for $n = 3$:
Again, we first perform the change of variable:

$$I = \int_0^2 f(x)dx = \frac{2-0}{2}\int_{-1}^{1} f\left(\frac{2-0}{2}(z-1)+2\right)dz = \int_{-1}^{1} f\left(z+1\right)dz.$$

and use the points and weights from the table:

$$\begin{aligned} I &\approx \sum_{i=1}^{3} w_i f(z_i + 1) \\ &= 0.\dot{5} \cdot f(-0.7745966692 + 1) + 0.\dot{8} \cdot f(0.0 + 1) + 0.\dot{5} \cdot f(0.7745966692 + 1) \\ &= 1.1393370 \end{aligned}$$

For the smooth integrand (part a), the Gauss Quadrature performs significantly better than the Trapezoidal and Newton-Cotes with the same amount of function evaluations. However, for non-smooth integrand (part b), the Trapezoidal rule wins over the Gauss Quadrature because it is able to obtain the exact solution of the piece-wise linear function. In general, one should observe that Gauss Quadrature has a significant drop in performance for non-smooth integrand.

b) In order to improve the performance of the Gauss Quadrature in the case of non-smooth integrands, i.e., non-continuous first order derivatives, one can split the domain of integration into sub-domains where the integrand is smooth and then apply the Gauss Quadrature in the sub-domains. In the second case, where $f(x) = 1 - |x-1|$, the integral should be split as follows:

$$\int_0^2 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x\,. \qquad (1)$$

Question 2: Adaptive Quadrature

Apply Adaptive Quadrature by hand, using the Trapezoid Rule with tolerance $tol = 0.05$ to approximate the integrals. Find the approximation error compared to the exact solution.

For both functions the approximation over the interval is performed using the trapezoidal quadrature rule. Let us denote the Trapezoid quadrature applied on the interval $[a, b]$ as $I_T[a, b]$:

$$I_T[a,b] = \frac{f(a) + f(b)}{2} * (b - a)$$

The error is estimated using Richardson extrapolation as:

$$\epsilon(h/2) = G(h/2) - G(h)$$

a) $f(x) = x^2$, $a_0 = 0$, $b_0 = 1$, $tol = 0.15$

We begin with $n = 1$ interval, $a_0 = 0$, $b_0 = 1$.

$$I_T[0,1] = \frac{0^2 + 1^2}{2}(1 - 0) = 1/2$$

We now split the interval into two sub-intervals at the midpoint $c = (a + b)/2 = 0.5$ and compute the integrals:

$$I_T[0, 0.5] = \frac{0^2 + (0.5)^2}{2}(0.5 - 0) = 1/16$$

$$I_T[0.5, 1] = \frac{(0.5)^2 + (1)^2}{2}(1 - 0.5) = 5/16$$

The error can be approximated as

$$\epsilon(h/2) = G(h/2) - G(h) = |(I_T[0, 0.5] + I_T[0.5, 1]) - I_T[0, 1]| = 1/8 = 0.125$$

We compare this error to our allowed relative tolerance

$$\epsilon(h/2) < tol \cdot \frac{h}{h_0}$$

$$0.125 < 0.15 \cdot \frac{1}{1} = 0.15$$

Thus we stop with the approximation $I_T = I_T[0, 0.5] + I_T[0.5, 1] = 3/8 = 0.375$ whereas the exact integral is $\int_0^1 x^2 dx = 1/3$. Thus the error is $\approx |0.375 - 1/3| \approx 1/24 < 0.15$.

b) $f(x) = \cos(x)$, $a_0 = 0$, $b_0 = \pi/2$, , $tol = 0.03$

We begin with $n = 1$ interval, $a_0 = 0$, $b_0 = \pi/2$.

$$I_T[0, \pi/2] = \frac{0 + 1}{2}(\pi/2) = \pi/4 \approx 0.78540$$

We now split the interval into two sub-intervals at the midpoint $c = (a+b)/2 = \pi/4$ and compute the integrals:

$$I_T[0, \pi/4] = \frac{1 + \cos(\pi/4)}{2}(\pi/4) \approx 0.67038$$

$$I_T[\pi/4, \pi/2] = \frac{\cos(\pi/4) + 0}{2}(\pi/4) \approx 0.27768$$

The error can be approximated as $|I_T[0, \pi/4] + I_T[\pi/4, \pi/2] - I_T[0, \pi/2]| = 0.16266 > 0.03 \cdot \frac{\pi/2}{\pi/2} = 0.03$. Thus we subdivide.

1. Interval $[0, \pi/4]$

$$I_T[0, \pi/4] = \frac{1 + \cos(\pi/4)}{2}(\pi/4) \approx 0.67038$$

$$I_T[0, \pi/8] = \frac{1 + \cos(\pi/8)}{2}(\pi/8) \approx 0.37775$$

$$I_T[\pi/8, \pi/4] = \frac{\cos(\pi/8) + \cos(\pi/4)}{2}(\pi/8) \approx 0.32024$$

The error can be approximated as $|I_T[0, \pi/8] + I_T[\pi/8, \pi/4] - I_T[0, \pi/4]| = 0.02762 > 0.03 \cdot \frac{\pi/4}{\pi/2} = 0.015$. Thus we subdivide.

(a) Interval $[0, \pi/8]$

$$I_T[0, \pi/8] = \frac{1 + \cos(\pi/8)}{2}(\pi/8) \approx 0.37775$$

$$I_T[0, \pi/16] = \frac{1 + \cos(\pi/16)}{2}(\pi/16) \approx 0.19446$$

$$I_T[\pi/16, \pi/8] = \frac{\cos(\pi/16) + \cos(\pi/8)}{2}(\pi/16) \approx 0.18699$$

The error can be approximated as $|I_T[0, \pi/16] + I_T[\pi/16, \pi/8] - I_T[0, \pi/8]| = 0.0037 < 0.03 \cdot \frac{\pi/8}{\pi/2} = 0.0075$. Thus we are done on the interval $[0, \pi/8]$ with the approximation $I_T[0, \pi/16] + I_T[\pi/16, \pi/8] \approx 0.38145$.

(b) Interval $[\pi/8, \pi/4]$

$$I_T[\pi/8, \pi/4] = \frac{\cos(\pi/8) + \cos(\pi/4)}{2}(\pi/8) \approx 0.32024$$

$$I_T[\pi/8, 3\pi/16] = \frac{\cos(\pi/8) + \cos(3\pi/16)}{2}(\pi/16) \approx 0.17233$$

$$I_T[]3\pi/16, \pi/4] = \frac{\cos(3\pi/8) + \cos(\pi/4)}{2}(\pi/16) \approx 0.15015$$

The error can be approximated as $|I_T[\pi/8, 3\pi/16] + I_T[3\pi/16, \pi/4] - I_T[\pi/8, \pi/4]| = 0.00314 < 0.03 \cdot \frac{\pi/8}{\pi/2} = 0.0075$. Thus we are done on the interval $[\pi/8, \pi/4]$ with the approximation $I_T[\pi/8, 3\pi/16] + I_T[3\pi/16, \pi/4] \approx 0.32338$.

2. Interval $[\pi/4, \pi/2]$

$$I_T[\pi/4, \pi/2] = \frac{\cos(\pi/4) + \cos(\pi/2)}{2}(\pi/4) \approx 0.27768$$
$$I_T[\pi/4, 3\pi/8] = \frac{\cos(\pi/4) + \cos(3\pi/8)}{2}(\pi/8) \approx 0.21398$$
$$I_T[3\pi/8, \pi/2] = \frac{\cos(3\pi/8) + \cos(\pi/2)}{2}(\pi/8) \approx 0.075140$$

The error can be approximated as $|I_T[\pi/4, 3\pi/8] + I_T[3\pi/8, \pi/2] - I_T[\pi/4, \pi/2]| = 0.01144 < 0.03 \cdot \frac{\pi/4}{\pi/2} = 0.015$. Thus we are done on the interval $[\pi/4, \pi/2]$ with the approximation $I_T[\pi/4, 3\pi/8] + I_T[3\pi/8, \pi/2] \approx 0.28912$.

There are no intervals left, and therefore the final approximation is $\approx 0.38145 + 0.32338 + 0.28912 = 0.99395$ whereas the exact integral is $\int_0^{\pi/2} \cos(x)dx = 1$. Thus the error is $\approx |0.99395 - 1| \approx 0.006 < 0.01$.

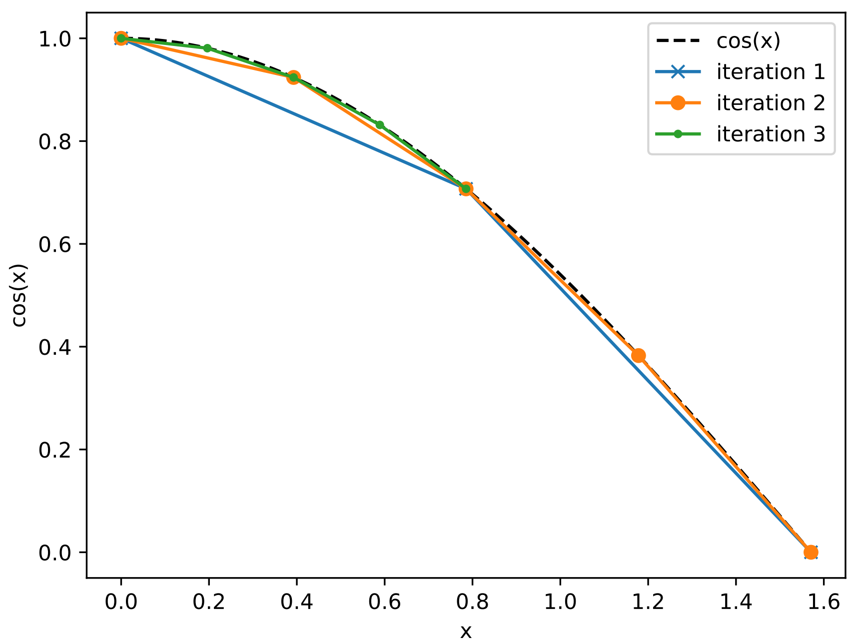

Figure 1: Quadrature points for the adaptive quadrature using the Trapezoidal rule for $f(x) = cos(x)$, on the interval $[0, \pi/2]$. This visualization shows that the left first sub-interval $[0, \pi/4]$ is less linear the right sub-interval $[\pi/4, \pi/2]$, thus requiring more iterations to reach the same tolerance (in this case one more).