Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Models, Algorithms and Data (MAD): Introduction to Computing

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Solution Set 7

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Question 1: Finite differences with Richardson extrapolation

a) To compute $G_2(h)$, we need to compute $G_0(h), G_0(h/2), G_0(h/4), G_1(h)$ and $G_1(h/2)$ as illustrated in Figure 1.

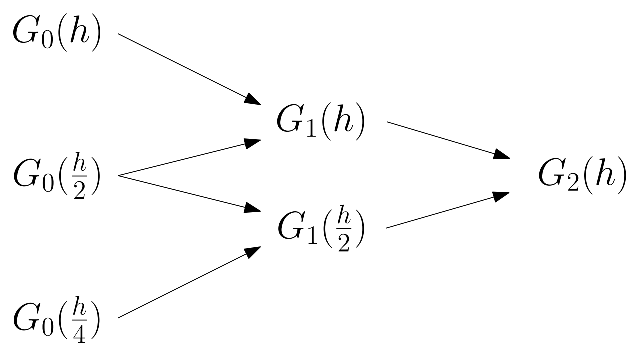

Figure 1: The recursive Richardson extrapolation scheme for the calculation of $G_2(h)$.

Letting $x = 0$ and $h = 0.4$ in the formula, the initial function evaluation is

$$G_0(h) = \frac{f(0.4) - f(0)}{0.4} = 2.22956,$$
$$G_0(h/2) = \frac{f(0.2) - f(0)}{0.2} = 2.10701,$$
$$G_0(h/4) = \frac{f(0.1) - f(0)}{0.1} = 2.05171.$$

Based on Richardson extrapolation, the first order can be computed as

$$G_1(h) = 2G_0(h/2) - G_0(h) = 1.98446,$$
$$G_1(h/2) = 2G_0(h/4) - G_0(h/2) = 1.99641.$$

The second order can be computed as

$$G_2(h) = \frac{4G_1(h/2) - G_1(h)}{3} = 2.00039.$$

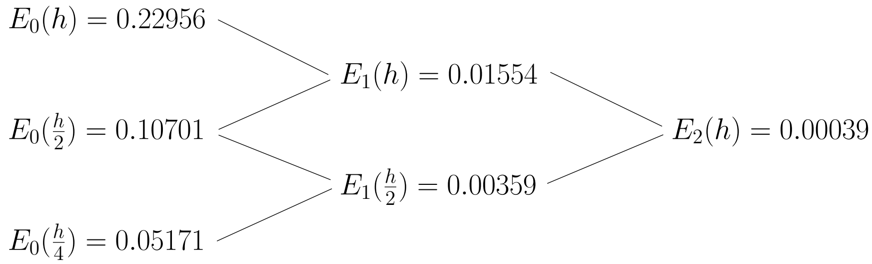

Figure 2: Error for each function evaluation using Richardson extrapolation.

b) The absolute value of the error for each term is summarized in Figure 2. We observe that in each iteration, using smaller h results in a higher accuracy. Moreover, the error is progressively reduced over iterations for the same h. In other words, we observe that

$$E_{n-1}(h) > E_{n-1}(h/2) > E_n(h).$$

Note: This relationship is what we expect from Romberg integration: The next iteration should give higher accuracy than the two parent approximations of the previous iteration. It is true that the order of accuracy increases because the Richardson extrapolation cancels out the low-order error terms. But order of accuracy is not the same as accuracy. If the constants in front of the higher-order error terms are large, it can happen that higher-order terms are actually larger than lower-order terms. Statements about the order of accuracy are asymptotic statements: They hold for h sufficiently small (for $h \to 0$). See the hints slides for an example of this.

Question 2: Pseudocode for Romberg integration

One possible pseudocode solution is given in Algorithm 1.

Algorithm 1 Romberg integration

Input:

function $f(x)$
interval boundaries a, b
number of iterations K

Output:

$I_K^1 = \texttt{integral}[K, 0]$ approximation to the integral $\int_a^b f(x)\,\mathrm{d}x$

Steps:

```
maxNumIntervals ← 2^K

// Precompute and store function evaluations
hmin ← (b − a)/maxNumIntervals
for i ← 0, ..., maxNumIntervals do
    fvalues[i] ← f(a + i * hmin)
end for

// Compute level 0 integrals
for r ← 0, ..., K do // refinement
    numIntervals ← 2^r
    step ← 2^(K−r) // step between two function evaluations for this refinement
    result ← 0
    for i ← step, 2 * step, 3 * step, ..., maxNumIntervals − step do
        result ← result + fvalues[i]
    end for
    // composite trapezoidal rule:
    integral[0, r] ← 0.5 (b−a)/numIntervals (fvalues[0] + fvalues[maxNumIntervals]
                                             +2 * result)
end for

//Advance to higher precision according to Romberg
for l ← 1, ..., K do //level
    for r ← 0, ..., K − l do // refinement
        integral[l, r] ← (4^l * integral[l−1, r+1] − integral[l−1, r]) / (4^l − 1)
    end for
end for
```

Question 3: Romberg integration

In the first step we calculate the necessary integrals on the first step.

$$I_0^1 = \frac{\pi}{2}(1+0) = \frac{\pi}{2} \approx 1.57080 \tag{1}$$

$$I_0^2 = \frac{\pi}{4}\left(1+2\left(\frac{2}{\pi}\right)+0\right) = \frac{4+\pi}{4} \approx 1.78540 \tag{2}$$

$$I_0^4 = \frac{\pi}{8}\left(1+2\left(\frac{2\sqrt{2}}{\pi}+\frac{2}{\pi}+\frac{2\sqrt{2}}{3\pi}\right)+0\right) = \frac{12+16\sqrt{2}+3\pi}{24} \approx 1.83551 \tag{3}$$

Now the the order of the approximation can be further increased with the Rhomberg integration.

$$I_1^1 = \frac{4I_0^2 - I_0^1}{3} = \frac{4+\pi-\frac{\pi}{2}}{3} = \frac{8+\pi}{6} \approx 1.85693 \tag{4}$$

$$I_1^2 = \frac{4I_0^4 - I_0^2}{3} = \frac{\frac{6+8\sqrt{2}}{3}+\frac{\pi}{2}-1-\frac{\pi}{4}}{3} = \frac{12+32\sqrt{2}+3\pi}{36} \approx 1.85221 \tag{5}$$

$$I_2^1 = \frac{16I_1^2 - I_1^1}{15} = \frac{4\frac{12+32\sqrt{2}+3\pi}{9}-\frac{8+\pi}{6}}{15} = \frac{72+256\sqrt{2}+21\pi}{270} \approx 1.85190 \tag{6}$$

In literature we find a better approximation of the so called Wilbraham-Gibbs constant of 1.85194. We can therefore see that our solution converges to the exact solution.