Models, Algorithms and Data (MAD): Introduction to Computing

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Prof. Dr. Jens Honore Walther
Dr. Georgios Arampatzis
ETH Zentrum, CLT
CH-8092 Zürich

Set 6

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In this exercise, you will use the Newton-Cotes formulas to derive the Simpson's rule for numerical integration and implement the trapezoidal and the Simpson's integration rules to solve an engineering problem.

Question 1: Simpson's rule from Newton-Cotes formulas

a) Use the Newton-Cotes formulas for $n = 2$ to compute the coefficients:

$$C_k^n = \frac{1}{b-a} \int_a^b l_k^n(x) dx, \quad k = 0, \ldots, n, \tag{1}$$

where $l_k^n(x)$ are Lagrange polynomials in interval $[a, b]$ of degree n:

$$l_k^n(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_n)}{(x_k - x_0)(x_k - x_1) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_n)}, \tag{2}$$

where x_i are equidistant points in $[a, b]$. For $n = 2$ that is: $x_0 = a,\ x_1 = (a+b)/2,\ x_2 = b$.

b) Using the computed coefficients C_k^n from (1), derive the resulting numerical integration rule using the Newton-Cotes formula:

$$I \approx (b - a) \sum_{k=0}^{n} C_k^n f(x_k). \tag{3}$$

Verify that you have obtained the so-called Simpson's rule:

$$I \approx \frac{f(a) + 4f((a + b)/2) + f(b)}{6}(b - a). \tag{4}$$

Question 2: Trapezoidal and Simpson's rule

a) Approximate the integral $I = \int\limits_0^\pi \sin(x)\, \mathrm{d}x$ numerically, making use of the trapezoidal and the Simpson rules. Use the derived sum notations for each rule from the lecture notes and the values from table 1.

b) Compare the obtained values to the true solution by computing the exact integral.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

Table 1: Function values of $\sin(x)$

Question 3: Order of convergence

Your colleague has made a study on the behaviour of errors with different integration schemes (rectangle, trapezoidal and Simpson's) over a domain of multiple integrals. Unfortunately he didn't label the graph correctly with the schemes.

a) How does the order of convergence of the different composite integration rules change when considered in a domain ([a,b]) with multiple intervals from the integration schemes on a single interval?

b) Can you assign the rectangle rule, the trapezoidal rule and the Simpson's rule to the three different plots? Explain your decision.

c) Under which circumstances can a higher order rule perform worse than a lower order rule?

d) How many more function evaluations are necessary for the trapezoidal rule to decrease the error by a factor of 1'000? How many more are necessary for the Simpson's rule?

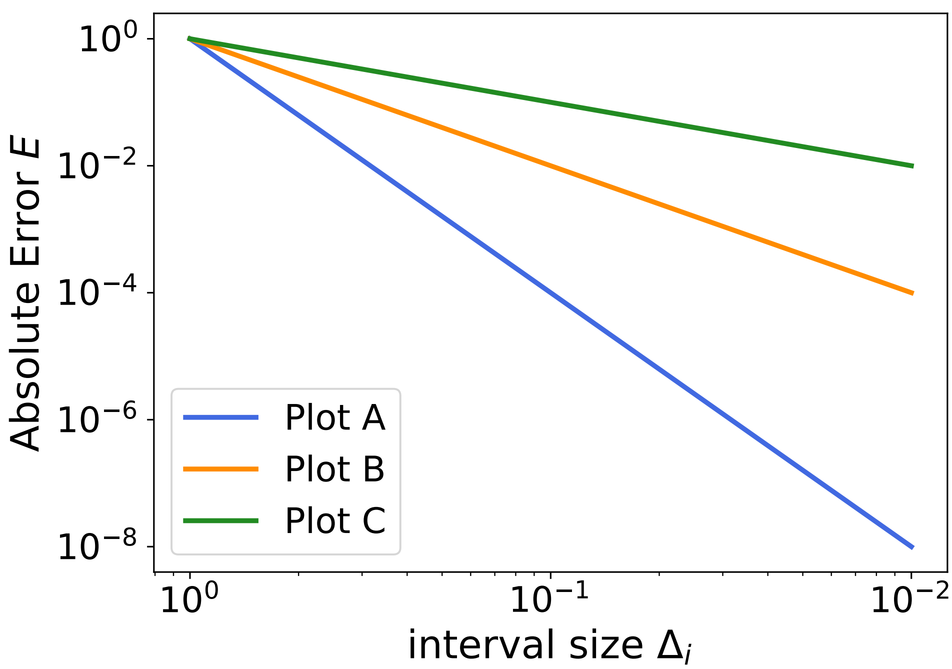

Figure 1: Graph of your colleague