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Set 5

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In this exercise, you will learn about fitting data using cubic splines. The first exercise is a numerical application of cubic splines with natural boundary conditions. In the second and third problems, you will derive the matrix equation for computing interpolating splines with 'clamped' and 'not-a-knot' boundary conditions respectively. In **Notebook 5** you will write a code to perform cubic spline interpolation using 4 nodes and different boundary conditions: natural, left side clamped, both sides clamped and 'not-a-knot'.

Question 1: Natural splines with uneven spacing

Let $f(x) = \sqrt{x+1}$. Given (0,1), (3,2), (8,3) construct a natural cubic spline. Rewrite the matrix equation Eq. 4.17 from the lecture notes for 3 points and derive the coefficients of the matrix system. Note that the nodes x_i's are here not equidistant, so the coefficients a_i, b_i, c_i, and d_i are not expected to be the same. Finally, write down the interpolating piecewise spline for the intervals defined by the given points.

Question 2: Cubic splines with clamped end conditions

Re-derive the matrix system shown in Eq. 4.17 in the lecture notes for interpolating cubic splines with clamped boundary conditions. The 'clamped' boundary condition refers to restricting the first derivatives at the end points to be zero, i.e., $f_1'(x_0), f_N'(x_N) = 0$.

Hint: Natural cubic splines ($f_0'', f_N'' = 0$) may give rise to non-zero derivatives at the end points. For clamped boundary conditions, we enforce the derivatives at the end points to be zero ($f_1'(x_0), f_N'(x_N) = 0$). To determine how this condition affects the second derivatives at $i = 0$ and $i = N$, we can use the relation

$$\text{for } x_{i-1} \le x \le x_i\text{:} \;\; f_i'(x) = f_i'' \left[\frac{(x - x_{i-1})^2}{2\Delta_i} - \frac{\Delta_i}{6} \right] - f_{i-1}'' \left[\frac{(x_i - x)^2}{2\Delta_i} - \frac{\Delta_i}{6} \right] + \frac{y_i - y_{i-1}}{\Delta_i}$$

and evaluate it at $x = x_0$ or $x = x_N$, respectively.

Question 3: Cubic splines with not-a-knot end conditions

Re-derive the matrix system shown in Eq. 4.17 in the lecture notes for interpolating cubic splines with the not-a-knot boundary condition. The 'not-a-knot' boundary condition assumes continuous third derivatives at the first and last two segments, i.e.

$$f_1'''(x_1) = f_2'''(x_1) \tag{1}$$

$$f_N'''(x_{N-1}) = f_{N-1}'''(x_{N-1}) \tag{2}$$

Write down the matrix system that you'd need to solve in order to construct the desired spline. Make sure to explicitly write out at least the first three and the last three rows of the system, in terms of $a_1, b_i, \cdots$ and any other variables you deem necessary.

<u>**Hint**</u>: The derivation can be performed either by including the nodes $0, N$ in the matrix system, or by excluding the equations at the boundaries and rewriting the matrix system for the interior nodes.