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Solution Set 5

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Note: For all following solutions, consider the notation for the solution of the tridiagonal matrix system from the lecture notes (Eq. 4.17): $a_i = f_i''$ for nodes with indices $0, \ldots, N$. The resulting segments will be with indices $i = 1, \ldots, N$, where $\Delta_i = x_i - x_{i-1}$.

Question 1: Natural splines with uneven spacing

Let $f(x) = \sqrt{x+1}$. Given $(x_0, y_0) = (0, 1)$, $(x_1, y_1) = (3, 2)$, $(x_2, y_2) = (8, 3)$ construct a natural cubic spline.

For any generic boundary conditions we must solve the following system:

$$\begin{bmatrix} B_0 & C_0 & 0 \\ A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \end{bmatrix} \cdot \begin{bmatrix} f_0'' \\ f_1'' \\ f_2'' \end{bmatrix} = \begin{bmatrix} D_0 \\ D_1 \\ D_2 \end{bmatrix}$$

In the case of natural spline we know that $f_0'' = 0$ and $f_2'' = 0$, from this we can fix $B_0 = 1$,$C_0 = 0$, $D_0 = 0$ and $A_2 = 0$,$B_2 = 1$, $D_2 = 0$. The values for the resulting coefficients are:

$$A_1 = \frac{\Delta_1}{6} = \frac{3}{6}\ ,\ B_1 = \frac{\Delta_1 + \Delta_2}{3} = \frac{8}{3}\ ,\ C_1 = \frac{\Delta_2}{6} = \frac{5}{6}$$

$$D_2 = \frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1} = \frac{3-2}{5} - \frac{2-1}{3} = \frac{-2}{15}$$

Resulting in:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{6} & \frac{8}{3} & \frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ f_1'' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{15} \\ 0 \end{bmatrix}$$

In the most general case you would need to invert the matrix. In this case we only need to solve for f_1'': $f_1'' = -\frac{1}{20}$

The values for the second derivatives then need to be replaced in Eq. 4.12 (with 4.14) from the lecture notes :

$$f_i(x) = f''_{i-1}\frac{(x_i - x)^3}{6\Delta_i} + f''_i\frac{(x - x_{i-1})^3}{6\Delta_i} + \left(\frac{y_i - y_{i-1}}{\Delta_i} - (f''_i - f''_{i-1})\frac{\Delta_i}{6}\right)(x - x_{i-1}) + \left(y_{i-1} - f''_{i-1}\frac{\Delta_i^2}{6}\right) \tag{1}$$

For the first interval the expression reads:

$$f_1(x) = f''_0\frac{(x_1 - x)^3}{6\Delta_1} + f''_1\frac{(x - x_0)^3}{6\Delta_1} + \left(\frac{y_1 - y_0}{\Delta_1} - (f''_1 - f''_0)\frac{\Delta_1}{6}\right)(x - x_0) + \left(y_0 - f''_0\frac{\Delta_1^2}{6}\right) \tag{2}$$

$$f_1(x) = \cancel{f''_0\frac{(x_1 - x)^3}{6\Delta_1}} + \frac{f''_1}{6\Delta_1}x^3 + \left(\frac{y_1 - y_0}{\Delta_1} - (f''_1 - \cancel{f''_0})\frac{\Delta_1}{6}\right)x + \left(y_0 - \cancel{f''_0\frac{\Delta_1^2}{6}}\right) \tag{3}$$

$$f_1(x) = 1 + \frac{43}{120}x - \frac{1}{360}x^3 \tag{4}$$

Following the same procedure for the second interval the following expression is derived:

$$f_2(x) = f''_1\frac{(x_2 - x)^3}{6\Delta_2} + f''_2\frac{(x - x_2)^3}{6\Delta_2} + \left(\frac{y_2 - y_1}{\Delta_2} - (f''_2 - f''_1)\frac{\Delta_2}{6}\right)(x - x_1) + \left(y_1 - f''_1\frac{\Delta_2^2}{6}\right) \tag{5}$$

$$f_2(x) = \frac{f''_1}{6\Delta_2}(8 - x)^3 + \cancel{f''_2\frac{(x - x_2)^3}{6\Delta_2}} + \left(\frac{y_2 - y_1}{\Delta_2} - (\cancel{f''_2} - f''_1)\frac{\Delta_2}{6}\right)(x - 3) + \left(y_1 - f''_1\frac{\Delta_2^2}{6}\right) \tag{6}$$

$$f_2(x) = \frac{265}{120} + \frac{19}{120}(x - 3) - \frac{1}{600}(8 - x)^3 \tag{7}$$

The total spline interpolation is expressed as:

$$f\ (x) = \begin{cases} f_1(x) = 1 + \frac{43}{120}x - \frac{1}{360}x^3 & 0 < x \le 3 \\ f_2(x) = \frac{265}{120} + \frac{19}{120}(x - 3) - \frac{1}{600}(8 - x)^3 & 3 < x \le 8 \end{cases}$$

Question 2: Cubic splines with clamped end conditions

In the case of clamped ends, all the equations except for the boundary nodes (f''_0 and f''_N) will be the same. In order to obtain new equations for f''_0 and f''_N recall the relation from the lecture notes, Eq. 4.11:

for $x_{i-1} \le x \le x_i$:

$$f'_i(x) = f''_i\left[\frac{(x - x_{i-1})^2}{2\Delta_i} - \frac{\Delta_i}{6}\right] - f''_{i-1}\left[\frac{(x_i - x)^2}{2\Delta_i} - \frac{\Delta_i}{6}\right] + \frac{y_i - y_{i-1}}{\Delta_i} \tag{8}$$

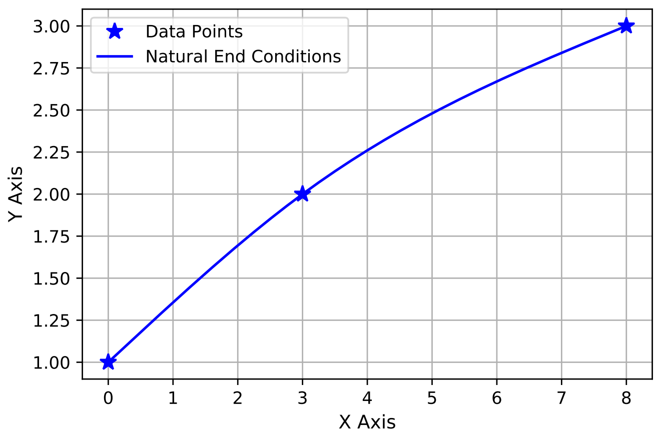

Figure 1: The resulting spline with natural boundary conditions. You can try and modify Notebook 5 to reproduce this result.

In the clamped boundary conditions, we enforce the values of f' to be zero at the end points. Having this in mind we evaluate Equation (8) at $x = x_0$ and $x = x_N$ for $i = 1$ and $i = N$, respectively.

For $x = x_0, i = 1$ we get:

$$0 = f_1'' \left[\frac{(x_0 - x_0)^2}{2\Delta_1} - \frac{\Delta_1}{6} \right] - f_0'' \left[\frac{(x_1 - x_0)^2}{2\Delta_1} - \frac{\Delta_1}{6} \right] + \frac{y_1 - y_0}{\Delta_1},$$

or

$$0 = -\frac{\Delta_1}{6} f_1'' - \left[\frac{\Delta_1}{2} - \frac{\Delta_1}{6} \right] f_0'' + \frac{y_1 - y_0}{\Delta_1},$$

and finally

$$\frac{\Delta_1}{3} f_0'' + \frac{\Delta_1}{6} f_1'' = \frac{y_1 - y_0}{\Delta_1} \Rightarrow$$
$$\tilde{B}_0 f_0'' + \tilde{C}_0 f_1'' = \tilde{D}_0.$$

Setting $x = x_N, i = N$ and doing similar computations we get a system for the f_N'':

$$\frac{\Delta_N}{6} f_{N-1}'' + \frac{\Delta_N}{3} f_N'' = -\frac{y_N - y_{N-1}}{\Delta_N} \Rightarrow$$
$$\tilde{A}_N f_{N-1}'' + \tilde{B}_N f_N'' = \tilde{D}_N.$$

The resulting system in matrix form will look as follows:

$$\begin{bmatrix} \tilde{B}_0 & \tilde{C}_0 & 0 & 0 & \dots & 0 \\ A_1 & B_1 & C_1 & 0 & \dots & 0 \\ 0 & A_2 & B_2 & C_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & 0 & 0 & 0 & \tilde{A}_N & \tilde{B}_N \end{bmatrix} \cdot \begin{bmatrix} f_0'' \\ f_1'' \\ f_2'' \\ \dots \\ f_{N-1}'' \\ f_N'' \end{bmatrix} = \begin{bmatrix} \tilde{D}_0 \\ D_1 \\ D_2 \\ \dots \\ D_{N-1} \\ \tilde{D}_N \end{bmatrix},$$

where the coefficients of the interior nodes are defined as:

$$A_i = \frac{\Delta_i}{6}\ ,\ B_i = \left(\frac{\Delta_i + \Delta_{i+1}}{3}\right)\ ,\ C_i = \frac{\Delta_{i+1}}{6}\ ,\ D_i = \frac{y_i - y_{i-1}}{\Delta_i} - \frac{y_{i-1} - y_{i-2}}{\Delta_{i-1}} \tag{9}$$

and the coefficients for the end points for the clamped boundary condition:

$$\tilde{B}_0 = \frac{\Delta_1}{3}, \tilde{C}_0 = \frac{\Delta_1}{6}, \qquad \tilde{D}_0 = \frac{y_1 - y_0}{\Delta_1} \tag{10}$$

$$\tilde{A}_N = \frac{\Delta_N}{6}, \tilde{B}_N = \frac{\Delta_N}{3}, \qquad \tilde{D}_N = -\frac{y_N - y_{N-1}}{\Delta_N} \tag{11}$$

Question 3: Cubic splines with 'not-a-knot' end conditions

The general system for nodes $x_i, i = 0, \ldots, N$ to solve for the second derivatives is:

$$\begin{bmatrix} \tilde{B}_0 & \tilde{C}_0 & 0 & 0 & ... & 0 \\ A_1 & B_1 & C_1 & 0 & ... & 0 \\ 0 & A_2 & B_2 & C_2 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & 0 & 0 & 0 & \tilde{A}_N & \tilde{B}_N \end{bmatrix} \cdot \begin{bmatrix} f''_0 \\ f''_1 \\ f''_2 \\ ... \\ f''_{N-1} \\ f''_N \end{bmatrix} = \begin{bmatrix} \tilde{D}_0 \\ D_1 \\ D_2 \\ ... \\ D_{N-1} \\ \tilde{D}_N \end{bmatrix}.$$

In order to extract the system for the 'not-a-knot' boundary condition, we need to extract the equations for the first and last nodes of the system. We work as follows:

The 'not-a-knot' condition assumes continuous third derivatives between the 2 first and last subintervals, i.e.:

$$f'''_1(x_1) = f'''_2(x_1) \tag{12}$$

$$f'''_N(x_{N-1}) = f'''_{N-1}(x_{N-1}) \tag{13}$$

$$\tag{14}$$

From the equation 4.10 of the lecture notes, and knowing that the second derivative is linear, the third derivative of the cubic spline in every segment is constant:

$$f'''_i(x) = \frac{f''_i - f''_{i-1}}{\Delta_i} = const., \tag{15}$$

where $\Delta_i = x_i - x_{i-1}$ Therefore, in order to determine the first cubic spline we get the equation for x_0:

$$\frac{f''_1 - f''_0}{\Delta_1} = \frac{f''_2 - f''_1}{\Delta_2} \tag{16}$$

plus the equation for x_1:

$$\Delta_1 f''_0 + 2(\Delta_1 + \Delta_2) f''_1 + \Delta_2 f''_2 = 6\left(\frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1}\right) \tag{17}$$

From Eq. 17:

$$f_2'' = \frac{6}{\Delta_2}\left(\frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1}\right) - \frac{\Delta_1}{\Delta_2} f_0'' - \frac{2(\Delta_1 + \Delta_2)}{\Delta_2} f_1'' \tag{18}$$

Therefore, Eq. 16 is written as

$$\begin{aligned}
-\Delta_2 f_0'' + (\Delta_2 + \Delta_1) f_1'' - \Delta_1 f_2'' &= 0 \\
-\Delta_2 f_0'' + (\Delta_2 + \Delta_1) f_1'' - \Delta_1 \left(\frac{6}{\Delta_2}\left(\frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1}\right) - \frac{\Delta_1}{\Delta_2} f_0'' - \frac{2(\Delta_1 + \Delta_2)}{\Delta_2} f_1''\right) &= 0 \\
(\Delta_1^2 - \Delta_2^2) f_0'' + (2\Delta_1^2 + 3\Delta_1\Delta_2 + \Delta_2^2) f_1'' = 6\Delta_1 \left(\frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1}\right) & \\
\tilde{B}_0 f_0'' + \tilde{C}_0 f_1'' &= \tilde{D}_0.
\end{aligned}$$

On the other end now for x_N:

$$\frac{f_N'' - f_{N-1}''}{\Delta_N} = \frac{f_{N-1}'' - f_{N-2}''}{\Delta_{N-1}} \tag{19}$$

plus the equation for x_{N-1}:

$$\Delta_{N-1} f_{N-2}'' + 2(\Delta_{N-1} + \Delta_N) f_{N-1}'' + \Delta_N f_N'' = 6\left(\frac{y_N - y_{N-1}}{\Delta_N} - \frac{y_{N-1} - y_{N-2}}{\Delta_{N-1}}\right) \tag{20}$$

We extract a relation for f_{N-2}'' via Eq. 20:

$$f_{N-2}'' = \frac{1}{\Delta_{N-1}}\left(6\left(\frac{y_N - y_{N-1}}{\Delta_N} - \frac{y_{N-1} - y_{N-2}}{\Delta_{N-1}}\right) - 2(\Delta_{N-1} + \Delta_N) f_{N-1}'' - \Delta_N f_N''\right), \tag{21}$$

and we use it to eliminate f_{N-2}'' from Eq. 19:

$$\begin{aligned}
-\Delta_N f_{N-2}'' + (\Delta_N + \Delta_{N-1}) f_{N-1}'' - \Delta_{N-1} f_N'' &= 0 \\
\frac{\Delta_N}{\Delta_{N-1}}\left(-6\left(\frac{y_N - y_{N-1}}{\Delta_N} - \frac{y_{N-1} - y_{N-2}}{\Delta_{N-1}}\right) + 2(\Delta_{N-1} + \Delta_N) f_{N-1}'' + \Delta_N f_N''\right) & \\
+(\Delta_N + \Delta_{N-1}) f_{N-1}'' - \Delta_{N-1} f_N'' &= 0 \\
(2\Delta_N^2 + 3\Delta_N\Delta_{N-1} + \Delta_{N-1}^2) f_{N-1}'' + (\Delta_N^2 - \Delta_{N-1}^2) f_N'' &= 6\Delta_N \left(\frac{y_N - y_{N-1}}{\Delta_N} - \frac{y_{N-1} - y_{N-2}}{\Delta_{N-1}}\right) \\
\tilde{A}_N f_{N-1}'' + \tilde{B}_N f_N'' &= \tilde{D}_N
\end{aligned}$$

According to this derivation, the general matrix system stays the same for the interior points and is modified only for the boundary points.

Another approach would be to write f_0'' and f_N'' through Eq. 16, 19:

$$f_0'' = \left(\frac{\Delta_1}{\Delta_2} + 1\right) f_1'' - \frac{\Delta_1}{\Delta_2} f_2'' \tag{22}$$

$$f_N'' = \left(\frac{\Delta_N}{\Delta_{N-1}} + 1\right) f_{N-1}'' - \frac{\Delta_N}{\Delta_{N-1}} f_{N-2}''. \tag{23}$$

$$\tag{24}$$

This way, we can eliminate f_0'' and f_N'' from the matrix system an rewrite the equations for the first and last interior nodes:

$$\left(\frac{\Delta_1^2}{\Delta_2} + 3\Delta_1 + 2\Delta_2\right) f_1'' + \left(\Delta_2 - \frac{\Delta_1^2}{\Delta_2}\right) f_2'' = 6\left(\frac{y_2 - y_1}{\Delta_2} - \frac{y_1 - y_0}{\Delta_1}\right) \tag{25}$$

$$\left(\Delta_{N-1} - \frac{\Delta_N^2}{\Delta_{N-1}}\right) f_{N-2}'' + \left(\frac{\Delta_N^2}{\Delta_{N-1}} + 3\Delta_N + 2\Delta_{N-1}\right) f_{N-1}'' = 6\left(\frac{y_N - y_{N-1}}{\Delta_N} - \frac{y_{N-1} - y_{N-2}}{\Delta_{N-1}}\right) \tag{26}$$

$$\tag{27}$$