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Solution Set 4

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Question 1: Lagrange interpolation pseudocode

Algorithm 1 Lagrange interpolation

Input:

arrays x, y
size N
point $\bar{x}$

Output:

$\bar{y}$ interpolation value at $\bar{x}$ of the data x, y

Steps:

```
ȳ ← 0
for k = 0, 1, ..., N − 1 do
    l ← 1
    for i = 0, 1, ..., N − 1 do
        if i ≠ k then
            l ← l ∗ (x̄ − x[i])/(x[k] − x[i])
        end if
    end for
    ȳ ← ȳ + l ∗ y[k]
end for
return ȳ
```

Question 2: Lagrange Interpolation

You sample the function $f(x) = \frac{-2}{3}x^3 + \frac{5}{3}x^2 + 4x$ at the points $\{x_i\} = \{0, 1, 3\}$ and wish to compute the Lagrange Interpolating polynomial for this dataset.

a) The Lagrange basis functions are given by:

$$\begin{aligned}
l_1(x) &= \frac{(x-1)(x-3)}{(0-1)(0-3)} \\
&= \frac{1}{3}(x^2 - 4x + 3) \\
l_2(x) &= \frac{(x-0)(x-3)}{(1-0)(1-3)} \\
&= \frac{-1}{2}(x^2 - 3x) \\
l_3(x) &= \frac{(x-0)(x-1)}{(3-0)(3-1)} \\
&= \frac{1}{6}(x^2 - x)
\end{aligned}$$

b) The full Lagrange interpolation function is achieved by taking linear combinations of the above computed basis functions weighted by their respective 'y' data points. First we compute the values of $f(x)$ at the points of interest:

$$\begin{aligned}
f(0) &= 0 \\
f(1) &= \frac{-2}{3} + \frac{5}{3} + 4 \\
&= 5 \\
f(3) &= \frac{-2}{3}3^3 + \frac{5}{3}3^2 + 12 \\
&= -18 + 15 + 12 \\
&= 9
\end{aligned}$$

The full Lagrange interpolating function is then given by:

$$\begin{aligned}
L(x) &= 0 \cdot l_1(x) + 5 \cdot l_2(x) + 9 \cdot l_3(x) \\
&= \frac{-5}{2}(x^2 - 3x) + \frac{3}{2}(x^2 - x) \\
&= 6x - x^2
\end{aligned}$$

c) Evaluating the Lagrange interpolation at $x = 2$ yields a result of 8, while evaluating the true objective function at $x = 2$ yields a result of $\frac{28}{3}$. The resulting interpolation error is $\frac{4}{3}$. Naturally, we shouldn't expect this interpolation error to be zero since we are modeling a cubic function with a quadratic. Had we sampled a fourth point, then the resulting Lagrange interpolating function would have been cubic instead of quadratic, and we would have had a perfect fit of the objective function and zero interpolation error.

Question 3: LSQ vs Lagrange

a) The goal is to predict the position at time $t = 5$, 1 second after the last measurement. The task is thus to extrapolate the existing data.

Two methods can be used:

- Data fitting using least squares

- Interpolation using Lagrange polynomials

Assumptions about the relation between x and t (ex. polynomial form) can be used when choosing the basis for LSQ fit or the number of points used for the Lagrange polynomials.

If you suspect your data to be noisy, data fitting is a better candidate as interpolation methods will pass exactly through your data points and take into account the noise. As you have seen in previous exercises, LSQ is robust to noise.

b) Based on the kinematic assumption, you know that the relation between time and position is 2^{nd} order polynomial. You thus choose t and t^2 as your basis. Let us note that 1 could also have been added as a basis if the measurements were taken relative to an initial position.

The goal is to find $\mathbf{w}$ in the linear system $A\mathbf{w} = \mathbf{y}$, where A is the matrix containing the basis functions and $\mathbf{y}$ the vector of measurements. The 2 basis functions used are t and t^2. The matrix A is defined as :

$$A = \begin{bmatrix} t_1 & t_1^2 \\ t_2 & t_1^2 \\ t_3 & t_1^2 \\ t_4 & t_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}$$

The system is solved using the normal equation to get the approximate solution $\mathbf{w}^\star$:

$$\mathbf{w}^\star = (A^T A)^{-1} A^T \mathbf{y}$$

Let us calculate explicitly the right-hand side matrix $H = A^T A$ and the left-hand side vector $Z = A^T\mathbf{y}$:

$$H = A^T A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$$

$$H^{-1} = \frac{1}{620} \begin{bmatrix} 354 & -100 \\ -100 & 30 \end{bmatrix}$$

$$Z = A^T\mathbf{y} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \times \begin{bmatrix} 12.5 \\ 30 \\ 52.5 \\ 80 \end{bmatrix} = \begin{bmatrix} 550 \\ 1885 \end{bmatrix}$$

To solve this linear system, it suffices to perform left-matrix-multiplication of the inverse matrix with both sides of the equation:

$$\begin{bmatrix} v_0 \\ \frac{1}{2}\alpha \end{bmatrix} = \mathbf{w}^\star = H^{-1}Z = \begin{bmatrix} 10 \\ 2.5 \end{bmatrix} \quad (1)$$

We get the values $\alpha = 5$, $v_0 = 10$. The position at $t = 5$ is therefore obtained from the resulting fitting function $x_{LSQ}(t) = 10t + 2.5t^2$, $x(5) = 112.5$m.

c) Replacing in the expression the already known values of t and x from the initial measurements, we can see that no noise is present in the measurements. Knowing that the relation is quadratic and that the data is without noise, we could have simply picked 2 time-points in the data and solved a system of 2 equations to find α and v_0.

d) Since we know that the relation is of 2nd order, we can apply Lagrange interpolation for N=3 data points. We select to interpolate using $t_1 = 1$, $t_2 = 2$, $t_3 = 4$:

$$l_1(t) = \frac{(t-2)(t-4)}{(1-2)(1-4)} = \frac{t^2 - 6t + 8}{3}$$

$$l_2(t) = \frac{(t-1)(t-4)}{(2-1)(2-4)} = \frac{t^2 - 5t + 4}{-2}$$

$$l_3(t) = \frac{(t-1)(t-2)}{(4-1)(4-2)} = \frac{t^2 - 3t + 2}{6}$$

$$\begin{aligned} x_{LI}(t) &= \sum_{i=1}^{3} y_i l_i(t) \\ &= 12.5 \cdot \frac{t^2 - 6t + 8}{3} + 30 \cdot \frac{t^2 - 5t + 4}{-4} + 8012 \cdot \frac{t^2 - 3t + 2}{6} \\ &= 2.5t^2 + 10t \end{aligned} \tag{2}$$

We get the values $\alpha = 5$, $v_0 = 10$, same as the LSQ fit from the previous question.

e) As long as there is no noise in the data, the interpolating polynomial will always be the same as above. With $N = 4$ data points we will get Lagrange basis polynomials of order $N - 1 = 3$. When constructing the interpolating function, higher order terms of the Lagrangian basis polynomials will cancel each other out, always resulting in the same quadratic polynomial.

f) As seen in previous exercise sessions the LSQ fit is robust to noise and would give a robust estimate of α and v_0.

Adding noise to the data would not change the Lagrange polynomials ($l_1(t)$,$l_2(t)$ and $l_3(t)$) as they are only functions of t. However the Lagrange interpolator ($x_{LI}(t) = \sum_{i=1}^{3} y_i l_i(t)$) is a direct function of y_i, where the noise contribution is added. Using Lagrange interpolation with N=3 data points, the resulting polynomial would pass exactly by the chosen data points and would not be robust to noise. Furthermore, if we used more data points (N=4), the higher order terms of the Lagrangian polynomials would no longer cancel each other out and would result in a 3^{rd} order polynomial.