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Set 3

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In this exercise, you will learn about Newton's method to solve nonlinear equations and systems of equations. In **Notebook 3.1** you will apply and compare Newton's and the Secant method. In **Notebook 3.2** you will apply Newton's method to solve a system of non-linear equations applied to a civil engineering problem.

Question 1: Newton's Method

a) Provide Newton's Method in pseudo-code. Therefore write down how you would in principle implement the ideas presented in the lecture. In order to get the idea regard the following pseudo-code computing the Fibonacci Series $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$ up the N'th term, while aborting if $F_n > n_{max}$.

Algorithm 1 Fibonacci Series

Input:

N, {number of elements to compute}
$n_{\max}$, {threshold to stop computation}

Output:

$\vec{F}$, {vector containing Fibonacci numbers}

Steps:

$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
$n \leftarrow 2$
while $n < N + 3$ **do**
    $F[n] \leftarrow F[n-1] + F[n-2]$
    if $F[n] > n_{\max}$ **then**
        break
    end if
    $n \leftarrow n + 1$
end while

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b) Estimate $\sqrt{34}$ up to the accuracy of 2 decimals using Newton's method. As initial guess use $x_0 = \sqrt{36} = 6$. To estimate the error of your approximation, assume that the exact solution is *not* known. How many iterations are needed?

An applied counterpart to this question can be found in Notebook 3.1.

Question 2: Convergence of Newton's Method

Newton's method should converge quadratically if certain conditions are respected.

a) What conditions are required for the quadratic convergence of Newton's method?

As seen in the lectures, the convergence rate r can be numerically estimated using

$$r \approx \frac{\log\left|\frac{e_{k+2}}{e_{k+1}}\right|}{\log\left|\frac{e_{k+1}}{e_k}\right|}$$

where $e_k = x_k - x^\star$.

b) In the previous exercise you have estimated $\sqrt{34}$ using Newton's method. Using the iterations below and the exact solution, numerically estimate the convergence rate for this problem. Does it match your expectations, why?

x_0	x_1	x_2	$x^\star$
6	$^{35}/_{6}$	$^{2449}/_{420}$	5.83095195

c) Repeat the calculation for the problem $f(x) = x^2 - 2x + 1$ using the iterations below. Does it match your expectations, why?

x_0	x_1	x_2	$x^\star$
2	1.5	1.25	1

Question 3: Convergence failure of Newton's Method

We have shown during the lecture that Newton's method is sensitive to the initial guess, which can decide to which root the method converges or if it converges at all. Another case for the convergence failure of Newton's method is when the derivative at the root $f'(x^\star)$ is not defined.

Apply Newton's method to the next three examples and report the different behaviours.

1. $f(x) = x^{1/3}$
2. $f(x) = x^{1/2}$
3. $f(x) = x^{4/3}$

Question 4: Newton's method for a system of nonlinear equations

Given the system of two nonlinear equations:

$$F(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2 \\ -x_1 + x_2^2 \end{pmatrix}$$

Find the Jacobian matrix $J(\mathbf{x})$ as well as the resulting iteration of Newton's method.