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Solution Set 3

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Question 1: Newton's Method

a) A pseudo-code of the Newton's Method is given in Algorithm 1.

Algorithm 1 Newton's method

Input:

x_0, {initial condition}
tol, {tolerance: stop if $\|x_k - x_{k-1}\| < \text{tol}$}
$k_{\max}$, {maximal number of iterations: stop if $k > k_{max}$}

Output:

x_k, {solution of $f(x_k) = 0$ within tolerance or if $k > k_{\max}$ reached}

Steps:

```
k ← 1
while k ≤ k_max do
    Calculate f(x_{k-1}) and f'(x_{k-1})
    Update x_k ← x_{k-1} − f(x_{k-1}) / f'(x_{k-1})
    if ||x_k − x_{k-1}|| < tol then
        break
    end if
    k ← k + 1
end while
```

b) $\sqrt{34}$ is a zero of a function $f(x) = x^2 - 34$. The derivative is $f'(x) = 2x$. One iteration of Newton's method

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} = x_{k-1} - \frac{(x_{k-1})^2 - 34}{2x_{k-1}} = \frac{x_{k-1}}{2} + \frac{17}{x_{k-1}}$$

Error estimate:

$$\epsilon_k = x_k - x_{k-1}$$

We iterate until $|\epsilon_k|$ reaches $1/100$. A convenient initial guess is $x_0 = \sqrt{36} = 6$.

$$x_1 = x_0/2 + 17/x_0 = 3 + 17/6 = 35/6, \quad \epsilon_1 = x_1 - x_0 = -1/6$$
$$x_2 = x_1/2 + 17/x_1 = 35/12 + 102/35 = 2449/420, \quad \epsilon_2 = x_2 - x_1 = -1/420$$

$|\epsilon_2| < 1/100$. We can stop at $x_2 = 2449/420 \approx 5.831$

A practical counterpart to this exercise can be found in Notebook 3.1.

Question 2: Convergence of Newton's Method

a) In order to prove the convergence of Newton's method, the derivative $f'(x_k)$ must be non zero for every iteration k. To converge quadratically the root needs to have a multiplicity of 1. For multiplicity of $m > 1$ the method converges linearly.

b) Using the iterations, the errors can be computed:

e_0	e_1	e_2
0.16904811	0.00238144	0.00000048

Replacing the errors in the equation for the convergence $r \approx \frac{\log\left|\frac{e_2}{e_1}\right|}{\log\left|\frac{e_1}{e_0}\right|} = 1.99$. The convergence rate is expected to be quadratic as the multiplicity of the root is 1. Furthermore, the derivative is non-zero for each x_k thus the result converges.

c) Using the iterations, the errors can be computed:

e_0	e_1	e_2
1	0.5	0.35

Using the formula above the convergence rate can be estimated numerically and gives $r \approx 1$. In this case, even though the function is differentiable everywhere the convergence is linear and not quadratic. This can be explained by the fact that $f(x)$ can be factored as $f(x) = x^2 - 2x + 1 = (x-1)^2$. The multiplicity of the root is $m = 2 > 1$, thus the quadratic convergence property is not obtained.

This can be further observed if the steps of the algorithm are written down explicitly. Substituting $f(x) = (x-1)^2$ and $f^{'}(x) = 2(x-1)$ in Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f^{'}(x_k)} = x_k - \frac{(x_k - 1)^2}{2(x_k - 1)} = x_k - \frac{x_k - 1}{2} = \frac{x_k + 1}{2} \tag{1}$$

As seen in the lecture notes, Newton's method converges quadratically if the following limit is finite:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = \lim_{k\to\infty} \frac{|f''(\xi_k)|}{2|f'(x_k)|} = \frac{|f''(x^\star)|}{2|f'(x^\star)|} = C < \infty\,, \tag{2}$$

In this case we have $x^\star = 1$ and thus

$$\lim_{x\to1} \frac{|f''(x)|}{2|f'(x)|} = \frac{|2|}{2|2(x-1)|} = \infty \tag{3}$$

This case shows that Newton's method converges linearly and not quadratically in case of root with multiplicity $m > 1$.

Question 3: Convergence failure of Newton's Method

One iteration of Newton's method is given as:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{4}$$

For each case, replacing $f(x)$ and $f^{'}(x)$ gives:

1. For $f(x) = x^{1/3}$, $f'(x) = \frac{x^{-2/3}}{3}$.
   The update rule becomes $x_{k+1} = x_k - \frac{x_k^{1/3}}{\frac{1}{3}x_k^{-2/3}} = x_k - 3x_k = -2x_k$. In this case, every iteration moves in absolute distance away from the root and the solution diverges.
2. $f(x) = x^{1/2}$, $f'(x) = \frac{x^{-1/2}}{2}$.
   The update rule becomes $x_{k+1} = x_k - \frac{x_k^{1/2}}{\frac{1}{2}x_k^{-1/2}} = x_k - 2x_k = -x_k$. If one starts from a positive value as x_0, then x_1 becomes $-x_0$, which is negative. However, we know that square root for a negative number is usually not defined (unless for an imaginary number) and therefore, not applicable for the Newton method here; If one starts from a negative value as x_0, we have the same issue of a negative number under square root already in the beginning.
3. $f(x) = x^{4/3}$, $f'(x) = \frac{4x^{1/3}}{3}$.
   The update rule becomes $x_{k+1} = x_k - \frac{x_k^{4/3}}{\frac{4}{3}x_k^{1/3}} = x_k - \frac{3}{4}x_k = \frac{1}{4}x_k$. In this case, the solution does converge as the derivative at $x^\star = 0$ is defined.

Question 4: Newton's method for a system of nonlinear equations

The Jacobian matrix is

$$J(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f_1(\mathbf{x})}{\partial x_1} & \dfrac{\partial f_1(\mathbf{x})}{\partial x_2} \\ \dfrac{\partial f_2(\mathbf{x})}{\partial x_1} & \dfrac{\partial f_2(\mathbf{x})}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & -1 \\ -1 & 2x_2 \end{pmatrix} \tag{5}$$

The inverse is

$$J^{-1}(\mathbf{x}) = \frac{1}{4x_1x_2 - 1} \begin{pmatrix} 2x_2 & 1 \\ 1 & 2x_1 \end{pmatrix} \tag{6}$$

Therefore, the iteration procedure is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{1}{4x_{k,1}x_{k,2} - 1} \begin{pmatrix} 2x_{k,2} & 1 \\ 1 & 2x_{k,1} \end{pmatrix} \begin{pmatrix} x_{k,1}^2 - x_{k,2} \\ x_{k,2}^2 - x_{k,1} \end{pmatrix}, \tag{7}$$

where $\mathbf{x}_k = (x_{k,1}, x_{k,2})^T$.