ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Computational Methods for Engineering Applications I

Spring semester 2016

Prof. P. Koumoutsakos, J. Šukys
ETH Zentrum, CLT F 12, E 11
CH-8092 Zürich

Exam

Issued: September 1, 2016, 9:00

Exam directives. In order to pass the exam, the following requirements have to be met:

- Clear your desk (no cell phones, cameras, etc.): on your desk you should have your Legi, your pen and your notes. We provide you with the necessary paper and the exam sheets.
- Read carefully the first two pages of the exam. Write your name and Legi-ID where requested. Before handing in the exam, **PUT YOUR SIGNATURE ON PAGE 2**.
- The personal summary consists of no more than 4 pages (2 sheets). The personal summary can be handwritten or machine-typed. In case it is machine-typed, the text has to be single-spaced and the font size has to be at least 8 pts. You are not allowed to bring a copy of somebody else's summary.
- The teaching assistants will give you the necessary paper sheets. You are not allowed to use any other paper sheets.
- You can answer in English or in German; the answers should be handwritten and clearly readable, written in blue or black - do NOT write anything in red or green. Only one answer per question is accepted. Invalid answers should be clearly crossed out. Whenever you write a C++-compatible pseudo code, include also the associated comments.
- For questions from 12 to 20, always use a new page for answering each new question (not for sub-questions!). On the top-right corner of every page write your complete name and Legi-ID. Unless otherwise noted in the question, you should hand-in your answers on paper!
- If something is disturbing you during the exam, or it is preventing you from peacefully solving the exam, please report it immediately to an assistant. Later notifications will not be accepted.
- You must hand in: the exam cover, the sheets with the exam questions and your solutions. The exam cannot be accepted if the cover sheet or the question sheets are not handed back.

Family Name:	
Name:	
Legi-ID:	

Question	Maximum score	Score	TA 1	TA 2
1	8			
2	5			
3	5			
4	10			
5	5			
6	5			
7	5			
8	5			
9	5			
10	4			
11	15			
12	20			
13	20			
14	20			
15	8			
16	20			
17	20			
18	10			
19	20			
20	30			
Total	240			

This exam sums up to 240 points. This is more than the score you are supposed to achieve; your ultimate goal is to reach 180 points.

With your signature you confirm that you have read the exam directives; you solved the exam without any unauthorized help and you wrote your answers following the outlined directives.

Signature:

Theory

Question 1: Lagrange Polynomials (8 points)

a) What is the highest degree of Lagrange polynomial you can uniquely fit to the data points shown in Fig. 1?

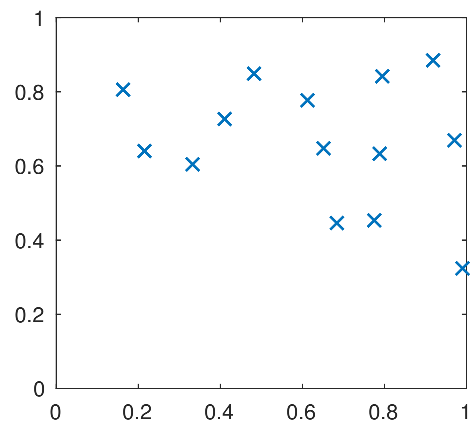

Figure 1

b) Can you think of a more appropriate interpolation method to fit the data points shown in Fig. 1? Explain your answer.

c) What is the highest order n of Newton-Cotes formula that you can use for computing the area under the curve formed by the points in Fig. 1? Why is this a good/bad choice? What would be the value of n if you were to use piecewise-parabolic segments for the computation?

Question 2: Data fit with Linear Least Squares (5 points)

Assume that you are given a set of data points obtained from shear flow experiments of Newtonian fluids. The available data are the shear stress, τ, and the shear rate, $\dot{\gamma}$. These quantities are physically related through the expression

$$\tau = \eta\dot{\gamma} \tag{1}$$

where η is the fluid viscosity. State whether the choice of using linear least squares to estimate the viscosity of a Newtonian fluid is justified and why.

Question 3: Graphical Linear Least Squares fit (5 points)

Draw an approximation of the linear least squares fit of a line (degree 1 polynomial) for the sets of data points provided in Figure 2.

Question 4: Cubic Splines multiple choice (10 points)

There is only 1 correct answer unless stated otherwise.

a) Cubic splines are guaranteed to have at maximum

1. 1^{st} order continuous derivatives

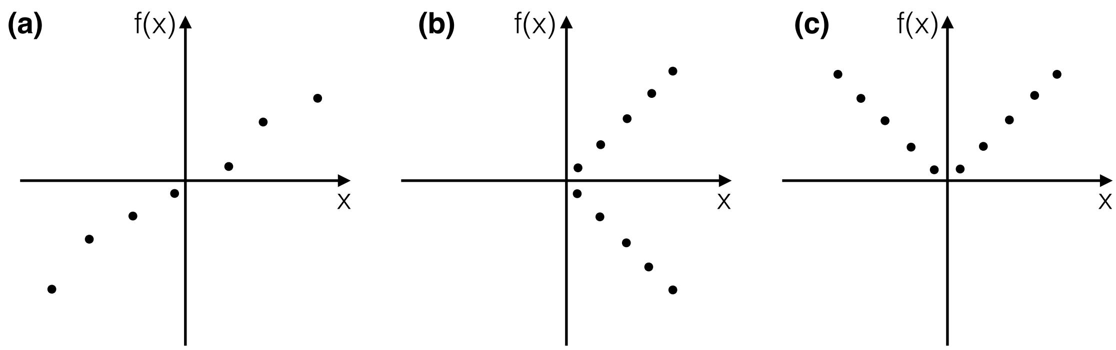

Figure 2: Data points for the least squares fit

2. 2^{nd} order continuous derivatives
3. 3^{rd} order continuous derivatives
4. do not necessarily have continuous derivatives

b) What is the minimum number of data points that when interpolated with Lagrange polynomials, will give a function that has a higher order than the Cubic splines interpolating function?

1. 3
2. 4
3. 5
4. infinite

c) Select ALL statements that are true:

1. In Cubic splines moving one point affects the entire fitting spline and thus the system of equations must be solved again.
2. High degree polynomials (e.g high order Lagrange polynomials) are highly oscillatory - a property that may not reflect the functions that are being approximated from the sampled points.

d) Draw cubic splines to fit the data points shown in Fig. 3 with the following boundary conditions (make sure to explain your answers): (a) Clamped left, Natural right (b) Natural left, clamped right

Question 5: NURBS (5 points)

a) List any three major concepts that are needed for NURBS interpolation.

b) Is it possible that the curve presented in Fig. 4 is a B-spline? Is it possible to be a NURBS? Completely justify your answer.

Question 6: Orthonormal polynomials (5 points)

What is the main advantage of using orthogonal basis functions for data fitting compared to Lagrange polynomials and Cubic Splines?

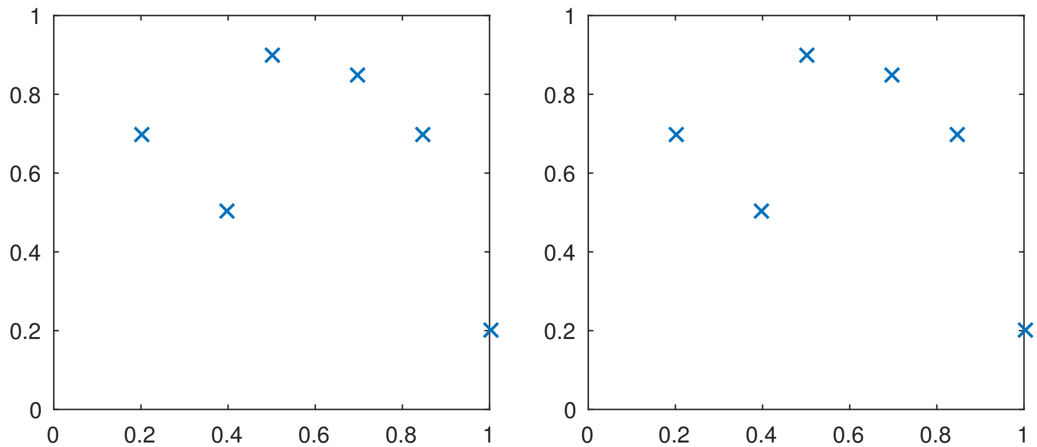

Figure 3: Figures with data points for cubic splines.

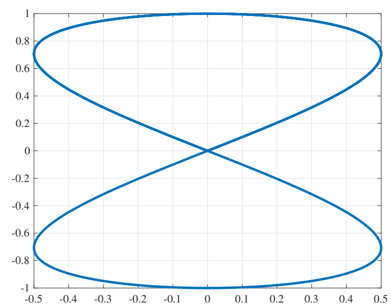

Figure 4: NURBS or B-spline?

Question 7: Graphical root estimation with Newton's method (5 points)

Find the solution of the non-linear function shown in Figure 5 graphically. In how many iterations would the method converge to the solution, if we accept the solution to be any point within the grey region? Draw the steps on the figure below.

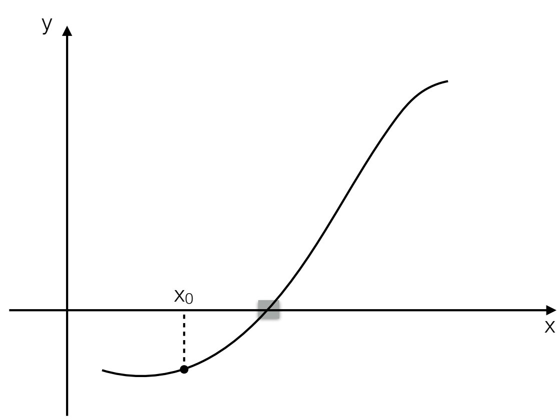

Figure 5: Graphical root estimation with Newton's method

Question 8: Advantages of higher order quadrature rules (5 points)

It is known that for $f \in C^2$, the quadrature error of the composite trapezoidal rule behaves like $O(n^{-2})$, for the number of quadrature points $n \to \infty$. Thus, the trapezoidal rule could be used to solve any numerical quadrature problem with C^2-integrand up to arbitrary accuracy.

Explain why other higher-order composite quadrature rules are relevant, nevertheless.

Question 9: Richardson extrapolation (5 points)

a) Richardson's extrapolation method aims to improve the approximation by combining approximations of different step sizes: True / False.

b) In the context of Richardson's extrapolation the approximation of a function is expressed as:

$$G(h) = G(0) + c_1 h + c_2 h^2 + c_3 h^3 + ... \tag{2}$$

$$G(h/2) = G(0) + \frac{1}{2} c_1 h + \frac{1}{4} c_2 h^2 + ... \tag{3}$$

and a better approximation of $G(0)$ can be obtained from:

$$G_1(h) = 2G(h/2) - G(h) = G + c_2' h^2 + c_3' h^3 + ... \tag{4}$$

Find an expression for c_3' in terms of c_3.

c) What is the leading error term for $G_1(h)$?

1. h
2. h^2
3. h^3
4. there is no error, $G_1(h)$ is the exact solution.

d) For the general approximating quantity $G_n(h)$, what is the leading error term?

1. h
2. h^n
3. h^{n+1}
4. h^{n-1}

Question 10: Adaptive Quadrature (4 points)

On the graphs shown in Fig. 6, qualitatively show how you expect the clustering of intervals to occur when using adaptive quadrature with the trapezoidal rule.

Question 11: Quadrature plots (15 points)

We consider three different functions on the interval $I = [0, 1]$:

$$\begin{aligned}
&\text{function A:} \quad f_A \in C^\infty(I)\,, \quad f_A \notin \mathcal{P}_k \ \forall\, k \in \mathbb{N}\,;\\
&\text{function B:} \quad f_B \in C^0(I)\,, \quad f_B \notin C^1(I)\,;\\
&\text{function C:} \quad f_C \in \mathcal{P}_{12}\,,
\end{aligned}$$

where $\mathcal{P}_k$ is the space of the polynomials of degree at most k defined on I. The following quadrature rules are applied to these functions:

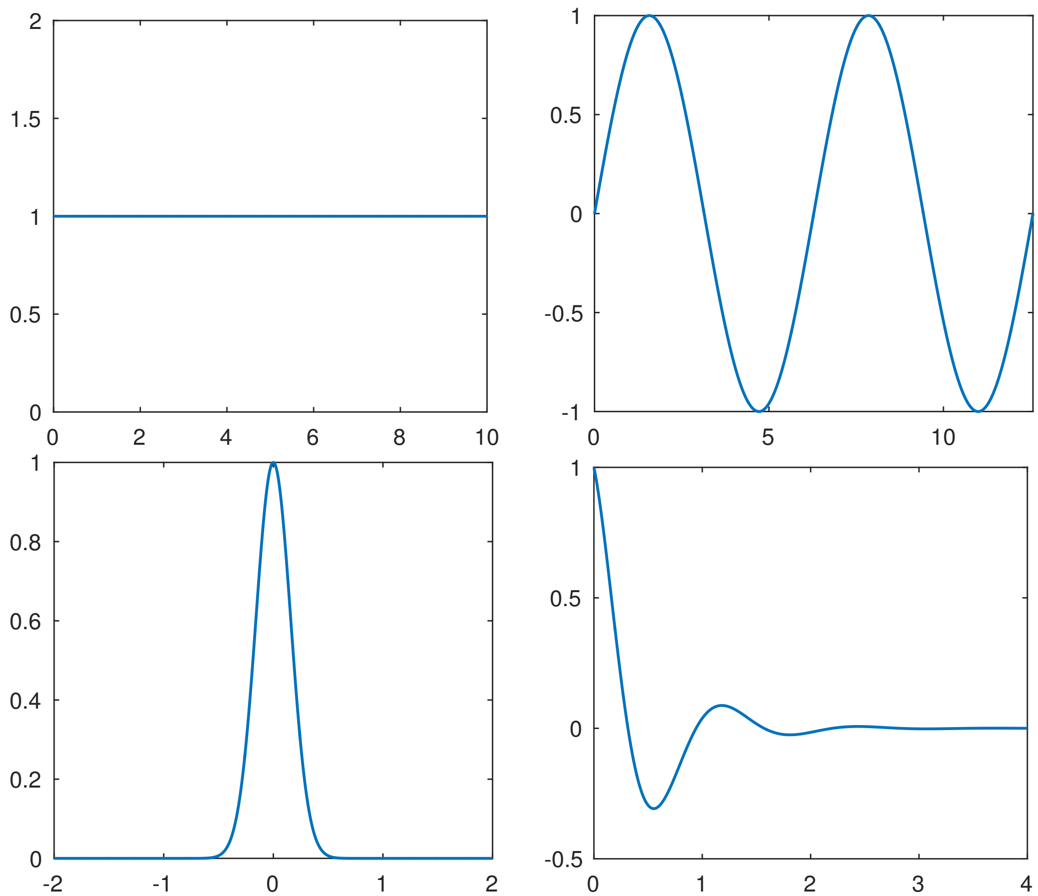

Figure 6: Draw clustering of intervals for adaptive quadrature with the trapezoidal rule.

- quadrature rule A, global Gauss quadrature;
- quadrature rule B, composite trapezoidal rule;
- quadrature rule C, composite 2-point Gauss quadrature.

The corresponding absolute values of the quadrature errors are plotted against the number of function evaluations in Figure 7. Notice that only the quadrature errors obtained with an even number of function evaluations are shown.

a) Match the three plots (plot #1, #2 and #3) with the three quadrature rules (quadrature rule A, B, and C). Justify your answer.

   Hint: notice the different axis scales in the plots.

b) The quadrature error curves for a particular function f_A, f_B and f_C are plotted in the same style (curve 1 as red line with small circles, curve 2 means the blue solid line, curve 3 is the black dashed line). Which curve corresponds to which function (f_A, f_B, f_C)? Justify your answer.

Numerical Problems

Question 12: Linear Least Squares (20 points)

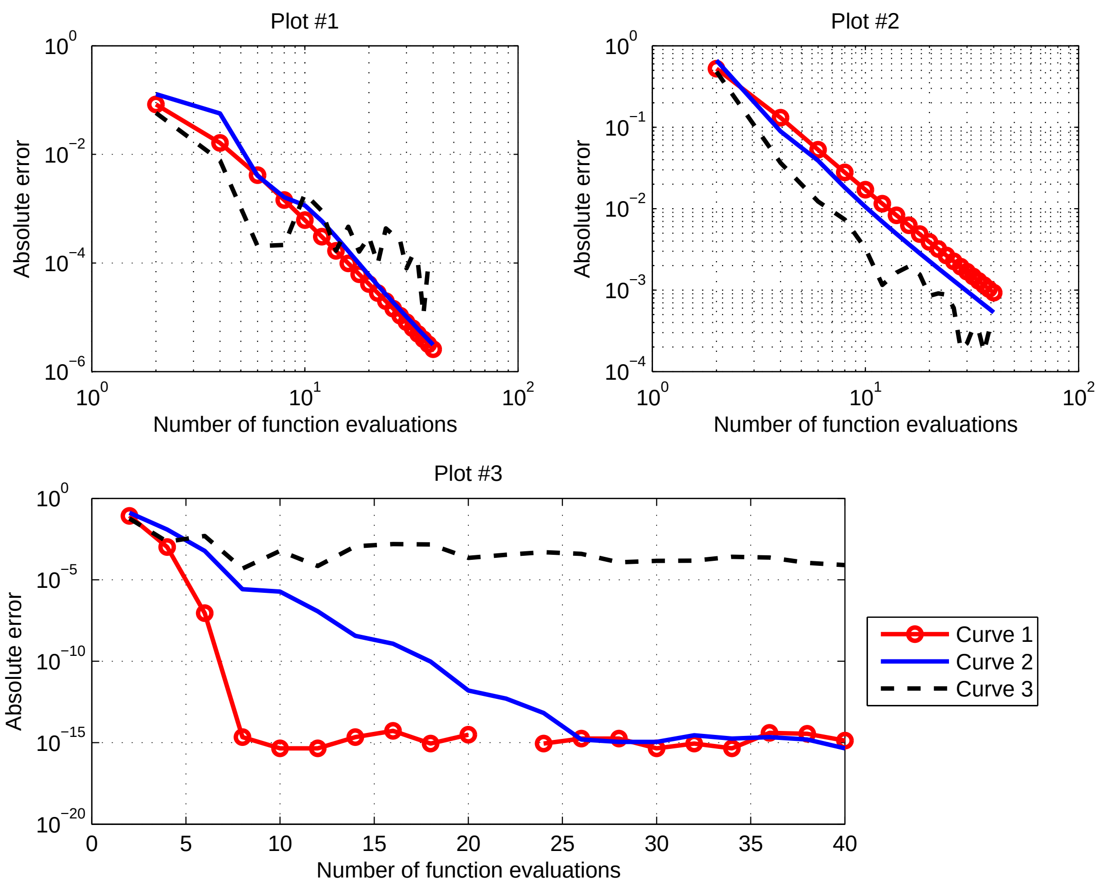

Figure 7: Quadrature convergence plots for different functions and different rules.

An experimentalist gives you a data set of observations $\{(x_i, y_i)\}_{i=1}^N$ where x_i corresponds to the input and y_i to the outcome of the experiment. He asks you to predict the output of the experiment for an input x^*. You decide to do it with Linear Least Squares and fit the function $f(x; a, b, c) = ax^2 + bx + c$ to the observations, i.e., $y_i \approx f(x_i; a, b, c)$, where a, b and c are the parameters that have to be determined.

a) Derive the system of linear equations that has to be solved in order to find the parameters a, b and c by doing the following:

   1. write the L_2 error of the approximation,
   2. take the derivative of the L_2 error with respect to a, b, c,
   3. set the derivatives to zero and derive the linear system of equations.

b) After fitting the data you change your mind and decide to fit a different function: $f(x; \alpha, \beta, \gamma) = \alpha e^{\beta x + \gamma x^2}$. Is it possible to find α, β, γ by using the matrix you derived in the previous question? Completely justify your answer.

Question 13: Finding roots with Newton's method (20 points)

a) Solve $e^x - x - 1 = 0$ by sketching an appropriate graph.

b) Solve $e^x = x + 2$ for the positive root using 3 iterations of the Newton method. You may leave your answer in terms of e and fractions if necessary. Pick appropriate initial guesses, and justify your choice. You have been given the following data: $e = 2.72$, $e^{-1} = 0.37$, $e^{-2} = 0.14$, and $e^2 = 7.39$.

Question 14: Newton-Cotes (20 points)

The Newton-Cotes formula for the approximation of a definite integral reads as,

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} c_k^n\, f(x_k)\,. \tag{5}$$

a) Derive the formulas for c_k^n. You do not need to evaluate (compute) the obtained integrals.

b) Prove that $\sum_{k=0}^{n} c_k^n = 1$ and $c_k^n = c_{n-k}^n$.

c) Describe potential problems with the Newton-Cotes formula when n is large. Can you give at least one alternative method that doesn't have this problem?

Question 15: Trapezoidal rule with derivatives (8 points)

Consider this quadrature rule:

$$\int_{-1}^{1} f(x)\,dx \approx a\,(f(-1) + f(1)) + b\,(f'(-1) - f'(1))\,.$$

If $a = 1$ and $b = 0$ it is a trapezoidal rule and the formula is exact for $f(x) = 1$ and $f(x) = x$. Find a and b which make the formula exact also for $f(x) = x^2$.

Question 16: Romberg integration (20 points)

a) Use the data provided in Fig. 8 to approximate $\int_1^5 y(x)dx$ using Romberg integration as accurately as possible. Keep your computations and final answer in fractions.

x	1	2	3	4	5
y	2	3	4	1	2

Figure 8: Data points used in Romberg integration.

Question 17: Multi-dimensional integration (20 points)

Figure 9 shows the surface $z = 0.1x^2 + 0.1y^3 + 4$.

a) Find the exact volume enclosed between the surface and the xy plane, for $x \in [-3, 3]$ and $y \in [-3, 3]$. (Hint: you can make your computation easier by exploiting the fact that z is an odd function of y.)

b) Compute the volume using the midpoint (rectangle) rule, with 3 intervals along each of the x and y axes. What is the absolute error for your numerical approximation with respect to the exact value of the integral?

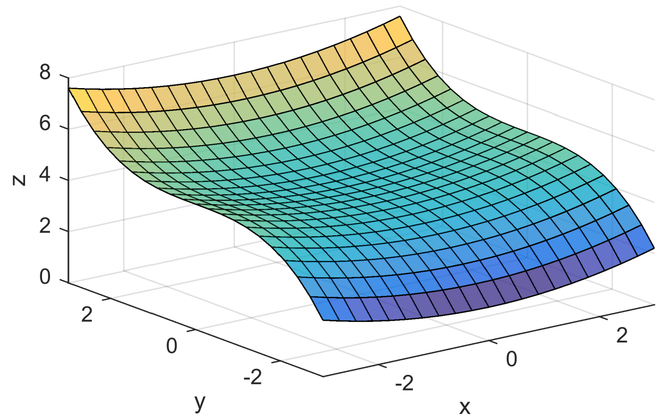

Figure 9

c) How is the error expected to change when you increase the number of function evaluations by a factor of 4?

Question 18: Monte Carlo integration (10 points)

The battery in the circuit shown in Fig. 10 supplies a constant voltage of $V = 10$ volts. The current I in the circuit depends on the voltage, and the resistance R according to relation $V = IR$. The instantaneous power delivered by the battery is $P = I^2R$. You are told that every time you switch on the circuit, R may randomly take on any positive value between 10Ω and 20Ω, following a given probability distribution $p(x)$. Furthermore, the value of R stays constant until you turn the circuit off.

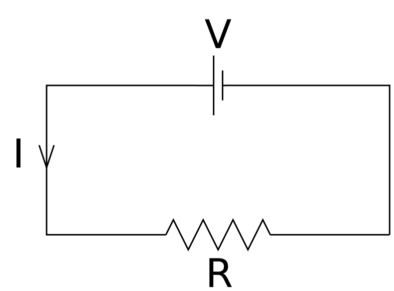

Figure 10

a) Write down the expression for P in terms of V and R.

b) Write down, in terms of V and $p(x)$, the expression for the expected (averaged over multiple switchings) power $\bar{P} = \mathbb{E}[P]$ delivered by the battery.

c) Approximate $\bar{P}$ using Monte Carlo method, given that 4 samples drawn from the given distribution $p(x)$ are as follows: $\{12, 17, 15, 16\}$.

Pseudo Codes

Question 19: Cross Validation (20 points)

a) What is the Cross Validation method and when is it used?

b) Write a pseudocode or a C++ code that implements the 2-fold Cross Validation method, i.e. the Cross Validation method that each time leaves out two points.

Question 20: Richardson + Romberg (30 points)

a) What are the respective orders of the piecewise polynomials constructed in each interval, when using the midpoint rule, the trapezoidal rule, and Simpson's rule?

b) Write down the formula for integrating a function $f(x)$ using the trapezoidal rule, with n intervals. Using this formula and Richardson's extrapolation, derive Simpson's rule.

c) Write down pseudocode to integrate $f(x)$ to an arbitrary order of accuracy J, using the composite trapezoidal rule and Romberg integration.

Good luck!