ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Models, Algorithms and Data (MAD): Introduction to Computing

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Set 5

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In this exercise, you will learn about the construction of B-splines and data fitting using Non-Uniform Rational B-Splines (NURBS). B-splines are basis functions of a given degree to define any other spline of the same degree. The construction is performed recursively as:

$$B_{i,0,t}(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$
$$B_{i,d,t}(x) = \frac{x - t_i}{t_{i+d} - t_i} B_{i,d-1,t}(x) + \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}} B_{i+1,d-1,t}(x). \tag{1}$$

By construction, the basis function $B_{i,d,t}(x)$ is a piecewise polynomial of degree d which is only non-zero for the range $t_i \leq x \leq t_{i+d+1}$. In addition, these basis functions have **continuous derivatives** up to degree $d-1$ (if all inner knots t_j are distinct). For the clamped case, use the convention 0/0=0. You will examine these properties of B-splines in this exercise.

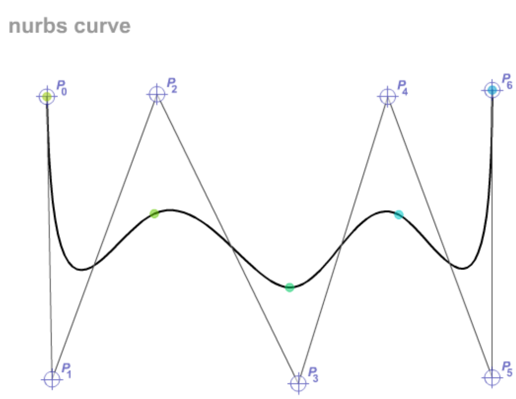

Figure 1: NURBS curve with cubic B-splines as basis functions. Source: geometrie.foretnik.net.

NURBS is a general framework for curve approximation. Given N data points $\vec{p}_i = \{x_i, y_i\}$ ($i = 1, \ldots, N$), the curve is now parametrized as $\vec{p}(s) = \{x(s), y(s)\}$ for $t_{d+1} \leq s \leq t_{N+1}$ based on B-splines as follows:

$$\vec{p}(s) = \sum_{i=1}^{N} R_{i,d,t}(s)\vec{p}_i, \text{ with } R_{i,d,t}(s) = \frac{B_{i,d,t}(s)w_i}{\sum_{j=1}^{N} B_{j,d,t}(s)w_j}, \tag{2}$$

where w_i is the weight for each basis function. In this case, we parametrize the x, y coordinates as $x(s), y(s)$ and then use for B-splines to approximate each one of the curves $x(s)$ and $y(s)$. The key idea is that we do not fit a function to the data but instead compute a linear combination of data points scaled by B-splines and user-specified weights.

Question 1: Construct B-splines

Based on Eq. (1), you need to derive the functional form of B-splines up to d=2 and plot the results. This is a pen-and-paper exercise.

a) Given 4 basis functions ($M = 4$), construct B-splines with degree $d = 1$. There are $M+d+1 = 6$ knots in this case. Derive the first two ($i = 1, 2$) functional form of B-splines using the knot vector with equal spacing $t = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Plot your results by hand.

b) Based on the derived functions in a), construct B-splines with degree $d = 2$. Use the same knot vector as in a). Derive the first ($i = 1$) functional form of B-splines, and plot your results by hand.

c) Check your plots to find the region where the B-spline functions are non-zero. Compute the first derivative of the B-spline computed in b), and check whether it is continuous at each knot.

d) Using the same settings, verify your results of questions 1a) and 1b) online at
http://geometrie.foretnik.net/files/NURBS-en.swf
Do you get the same plots as shown on the website?

Question 2: Compute and draw a "hidden" NURBS spline

In this exercise, you are asked to compute and draw (using pen and paper) a third-order ($d = 2$) "hidden" spline which is described by the location of the control points, weights, and knots. In particular, we set the control points $p_i = (x_i, y_i)$ to lie in the corners and in the mid-points of the edges of a square:

$$\begin{aligned}
\vec{p}_1 &= (-1, 0),\\
\vec{p}_2 &= (-1, 1),\\
\vec{p}_3 &= (0, 1),\\
\vec{p}_4 &= (1, 1),\\
\vec{p}_5 &= (1, 0),\\
\vec{p}_6 &= (1, -1),\\
\vec{p}_7 &= (0, -1),\\
\vec{p}_8 &= (-1, -1),\\
\vec{p}_9 &= (-1, 0).
\end{aligned} \tag{3}$$

For an illustration, see Figure 2 - notice, that the figure is generated using geometrie.foretnik.net, where indexing starts from 0 instead of 1 as is the case here and in the lecture notes.

Notice that the first and the last control points coincide (i.e. $\vec{p}_1 = \vec{p}_9$). Additionally, the weights are also given (see Figure 2):

$$w_1 = w_3 = w_5 = w_7 = w_9 = 1, \qquad w_2 = w_4 = w_6 = w_8 = \sqrt{2}/2. \tag{4}$$

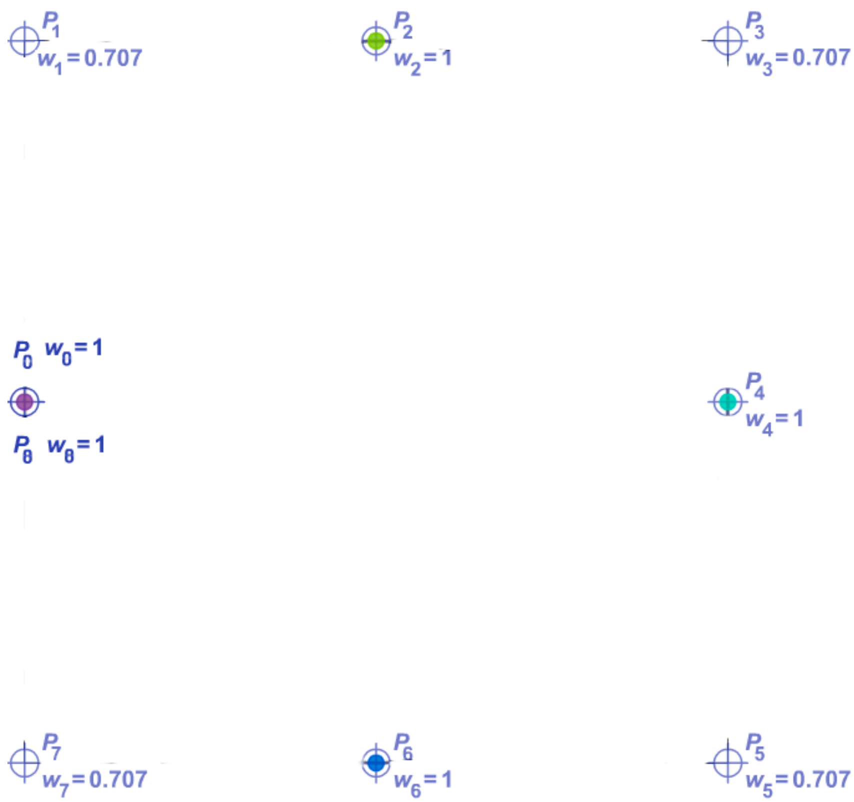

Figure 2: Locations of the control points and weights for the "hidden" spline. Source: geometrie.foretnik.net.

Finally, the knot vector is given by (notice clamped (repeated) knots)

$$t = \left(0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1\right). \tag{5}$$

a) For this particular setting, how many non-trivial (i.e. length > 0) intervals $t_i \leq s \leq t_{i+1}$ do we have? Hint: careful with repeating knots.

b) Now we reduce to only the first three (i.e. set $M = 3$) control points $\vec{p}_1, \vec{p}_2, \vec{p}_3$ and use the knot vector

$$t = (0, 0, 0, 1, 1, 1). \tag{6}$$

Using (1), compute the resulting B-splines for such three control points.
Hint: for repeated knot points, use the convention 0/0=0.

c) Using the B-splines from the previous sub-question, compute the NURBS spline $\vec{p}(s) = \{x(s), y(s)\}$ given by (2).
Hint: remember to use the specified weights.

d) On paper, draw the control points and the computed spline. Direct drawing of the graphs for the rational polynomials is very complicated; to avoid this, in this particular case you could firstly compute the following quantity:

$$\|\vec{p}(s)\| = \|(x(s), y(s))\| = \sqrt{x(s)^2 + y(s)^2} = ? \tag{7}$$

Which curve (think simple) satisfies such relation?

e) Without computing anything, can you guess what would be the shape of the entire spline (i.e. if all control points $\vec{p}_1, \ldots, \vec{p}_9$ were included)?

In that situation: what happens, if we move a single control point? Which part of the curve do you expect to change?

Verify your expectation online at http://geometrie.foretnik.net/files/NURBS-en.swf