Models, Algorithms and Data (MAD): Introduction to Computing

Spring semester 2019

Prof. Dr. Jens Honore Walther
Dr. Julija Zavadlav
ETH Zentrum, CLT
CH-8092 Zürich

Set 3

Issued: 08.03.2019; Due: 17.03.2019

In this exercise, you will learn about solving nonlinear system of equations with Newton's method.

Review of Newton's method: You are given a system of N nonlinear equations $f_i(\vec{x})$ ($i = 1, \ldots, N$) where $\vec{x} = (x_1, \ldots, x_N)$ is a vector of N unknowns. We write the system of equations as $\vec{F}(\vec{x}) = \vec{0}$ and we define the $N \times N$ Jacobian matrix $J(\vec{x})$ with elements $J(\vec{x})_{i,j} = \partial f_i(\vec{x})/\partial x_j$. Newton's method starts with an initial approximation $\vec{x}^{(0)}$ and iteratively improves the approximation by computing new approximations $\vec{x}^{(k)}$ as in Algorithm 1. The method converges when the difference $\|\vec{y}^{(k-1)}\| = \|\vec{x}^{(k)} - \vec{x}^{(k-1)}\|$ between the current and the previous approximation is lower than a user-specified tolerance.

Algorithm 1 Newton's method

Input:
- $\vec{x}^{(0)}$, {vector of length N with initial approximation}
- tol, {tolerance: stop if $\|\vec{x}^{(k)} - \vec{x}^{(k-1)}\| < tol$}
- k_{max}, {maximal number of iterations: stop if $k > k_{max}$}

Output:
- $\vec{x}^{(k)}$, {solution of $\vec{F}(\vec{x}^{(k)}) = \vec{0}$ within tolerance tol} (or a message if $k > k_{max}$ reached)

Steps:
```
k ← 1
while k ≤ k_max do
    Calculate F(x^(k-1)) and N × N matrix J(x^(k-1))
    Solve the N × N linear system J(x^(k-1)) y^(k-1) = −F(x^(k-1))
    x^(k) ← x^(k-1) + y^(k-1)
    if ||y^(k-1)|| < tol then
        break
    end if
    k ← k + 1
end while
```

Question 1: Pressure to sink object

Your goal in this exercise is to make use of Newton's method to solve the architects' problem, i.e. formulate and solve a system of nonlinear equations trying to estimate the size of the bridge-foundations (see Figure 1) such that the under a given load the bridge will not sink more than a certain depth.

Figure 1: Bridge supported by regularly spaced foundations.
http://www.bristol.ac.uk/civilengineering/bridges/Pages/HowtoreadabridgeFoundations.html

The pressure required to sink a large, heavy object in the soft homogeneous soil, that lies above the hard-base soil, can be predicted by the pressure required to sink smaller objects in the same soil. The bridge-foundations can be modeled as circular plates. The pressure p required to sink a circular plate of radius r in the soft soil to a certain depth d can be approximated by an expression:

$$p(r) = k_1 e^{k_2 r} + k_3 r$$

where $k_1,\ k_2 > 0$, and k_3 depend on d and the consistency of the soil but not on the radius of the plate.

a) You have the following data: a pressure of 100 N/m^2 is required to sink a plate of radius $r = 0.1$ m to depth $d = 1$ m, whereas a plate of radius $r = 0.2$ m requires a pressure of 120 N/m^2 and a plate of radius $r = 0.3$ m requires a pressure of 150 N/m^2 to get sunk to the same depth d. Formulate the system of equations to be solved for the coefficients $k_1,\ k_2$, and k_3 and the Jacobian matrix of the resulting system.

b) Implement Newton's method to find $k_1,\ k_2$, and k_3. Set tolerance for convergence to 10^{-5}.

c) Using the results from (b), predict the radius of the bridge foundations that will prevent the bridge from sinking more than 1 m. You are told by the architects that each foundation must support a load of 50000 N.

   Note: Load is not the same as pressure!

d) Solve subquestion (c) graphically.