# High Performance Computing for Science and Engineering II

ETH zürich

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# Set 02 - Bayesian inference

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## Question 1: Linear Model

You are given the linear regression model that describes the relation between variables x, y.

$$y = \alpha + \beta x + \epsilon,$$

where $\alpha$ and $\beta$ are the regression parameters, y is the output quantity of interest (QoI) of the system, x is the input variable and $\epsilon$ is a term accounting for model and measurement errors. For all following subquestions consider that the prior uncertainty for the parameter $\beta$ is quantified by a uniform distribution with large enough bounds. The regression parameter $\alpha$ is not considered uncertain. The model error is quantified by a Gaussian distribution $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

a) First consider the model, where $\alpha = 0$, given one measurement data point, $D = \{x_0, y_0\}$:

$$y = \beta x + \epsilon$$

Find the posterior uncertainty in the model parameter $\beta$ (i.e. determine the posterior distribution, the negative log-likelihood function and the most probable value (MPV) of $\beta$).

b) Second, using the same model as above, now consider a dataset of three output points, with the same input value $x_0$, $D = \{y_0, y_0/2, 2y_0\}$. Each observation is therefore fitted by the model:

$$y_i = \beta x_i + \epsilon_i,$$

where $i = 1...N$ and $\epsilon_i$ are independent and identically distributed error terms, $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.
Find the posterior uncertainty in the model parameter $\beta$, using the new dataset.

c) Here consider the complete regression model, where only $\beta$ is an uncertain model parameter. You are given a dataset of N measurement points $D = \{X, Y\}$, where $X = \{x_1, ..., x_N\}$ and $Y = \{y_1, ..., y_N\}$. Each observation is, therefore, fitted by the model

$$y_i = \alpha + \beta x_i + \epsilon_i,$$

where $i = 1...N$ and $\epsilon_i$ are independent and identically distributed error terms, $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.
Find the posterior distribution, negative log-likelihood function and the most probable value (MPV) of the model parameter $\beta$, given the dataset $D$.

## Question 2: Non-Linear Model

Consider the mathematical model of a falling object with mass $m$, acceleration of gravity $g$ and air resistance force $F_{res} = -m\alpha v^2$, where $\alpha$ is the air resistance coefficient. Using Newton's law, the equation of motion of the falling object is:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - m\alpha v^2 \tag{1}$$

The solution for the velocity obtained from the nonlinear differential equation (1) is:

$$v(t) = v_\infty \tanh\left(\frac{g(t-t_0)}{v_\infty}\right), \tag{2}$$

where $v_\infty = \sqrt{g/\alpha}$ and $t_0$ is the initial time. Integrating the velocity $v(t)$ with respect to time, the solution for the vertical displacement $x$ of the falling object is finally obtained as

$$x(t) = \frac{1}{\alpha}\ln\cosh\left(\sqrt{g\alpha}(t-t_0)\right). \tag{3}$$

Measurements for the position of the falling object are obtained by a digital camera making snapshots with an interval of $\Delta t$ sec. Given the observation data $D = \{\hat{X}_1, ..., \hat{X}_N\}$ of the location of the falling object at time instances $t = \{\Delta t, ..., N\Delta t\}$, respectively, we are interested in estimating the uncertainty of the parameter $\alpha$ of the system given the value of the variance $\sigma^2$. Note that the measurements and the model predictions satisfy the model error equation

$$\hat{X}_k = x(k\Delta t) + E_k, \tag{4}$$

where the measurement error terms $E_k$ are independent identically distributed (i.i.d.) and follow the zero-mean Gaussian distribution $\mathcal{N}(0, \sigma^2)$.

Assume a uniform prior for $\alpha$ and derive the expressions for the

1) Posterior PDF (probablity density function) $p(\alpha|D,\sigma)$,
2) The negative log-likelihood function $L(\alpha) = -\ln p(\alpha|D,\sigma)$,
3) Consider a simple case of only one measurement $\hat{X}_1 = 1[m]$ taken 1 s after the beginning of the fall. Assuming $g = 9.8\,[m/s^2]$, show that $\hat{\alpha} = 8.366\,[1/m]$ is the most probable value of the air resistance coefficient (the value which maximizes the posterior PDF).
   Hint: You do not need to solve the resulting equation. Only show that the given value is indeed a minimum.