Sample Final Exam

Bayesian Inference

Problem 1.

The posterior distribution of the parameters of a model is given by

$$p(\theta_1,\theta_2 \mid D,I) \propto \exp\left[-\frac{1}{2}\left(\theta_1^2+\theta_2^2+\theta_1\theta_2-\theta_1-2\theta_2+1\right)\right]$$

Find the uncertainty region and plot it in the two-dimensional parameter space (θ_1,θ_2).

<u>Hint</u>: Need to find the most probable point, the Hessian, the covariance matrix and then <u>clearly plot the contour plots</u> of the posterior distribution in the two-dimensional parameter space, indicate the principal direction of the ellipsoid, as well as the length of the uncertainty along the principal axes of the ellipsoid.

Problem 2.

Consider the mathematical model of a physical process/system represented by the equation

$$Y = aX_1 + E$$

where X_1 is the uncertain parameter of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0,S)$, where $S \in R$ is known. Given the three independent measurements $\hat{Y} = \{y_0, 2y_0, 3y_0\}$

a. Find the posterior uncertainty in the model parameter X_1. The prior uncertainty in X_1 is quantified by
   i. a uniform distribution with very large bounds
   ii. a Gaussian distribution with mean μ and standard deviation σ
b. For the case (ii), find the uncertainty in the output quantity of interest

$$Z = bY + \eta$$

where the error term η is a Gaussian distribution with mean zero and variance S_0.

c. For the case (ii), find the posterior probability that Z exceeds $6by_0$.

Problem 3.

Consider the mathematical model of a physical process/system represented by the equation

$$Y = a\log(X_1) + E$$

where X_1 is the uncertain parameter of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0, S)$, where $S \in R$ is known. Given the independent measurements $\hat{Y} = \{y_0, 2y_0, 3y_0\}$,

a. Find the posterior uncertainty in the model parameter X_1 using Bayesian central limit theorem. The prior uncertainty in X_1 is quantified by a Gaussian distribution with mean μ and standard deviation σ.
b. Approximate the uncertainty in the output quantity of interest

$$Z = bY + \eta$$

where the error term η is a Gaussian distribution with mean zero and variance S_0.

Markov Chain Monte Carlo

Problem 4.

The posterior probability density function of a set of two parameters $\underline{\theta} = (\theta_1, \theta_2)^T$ is Gaussian with mean $\underline{0}$ and diagonal covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Let $\underline{\theta}^{(j)}$ be the current sample in the Markov Chain Monte Carlo algorithm generated using a Metropolis-Hasting algorithm. Following Metropolis-Hasting algorithm, let $\underline{\xi}$ be the candidate sample drawn from a uniform distribution centered at the current sample $\underline{\theta}^{(j)}$. Let $\underline{\theta}^{(j)} = (1,0)^T$. Let also $\underline{\xi} \sim U([0,3],[0,1])$, is drawn from a uniform distribution with bounds $[0,3]$ for the first component ξ_1 and $[0,1]$ for the first component ξ_2.

1. If the sample $\underline{\xi} = (2,1)^T$, find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (2,1)^T$
2. If the sample $\underline{\xi} = (0,0)^T$, find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (0,0)^T$