Prof. Petros Koumoutsakos
CSE Lab, ETH Zürich
www.cse-lab.ethz.ch

Advanced Computational Science
SS 2009

Serie 10 Ausgabe: May 5, 2009 Abgabe: May 12, 2009

Guidelines

▷ Submission: Submit a written or typed report with your answers and graphics.

▷ Code: Send your source code (and makefile) to the TA of your session. Use "[ACS] Homework10" in the subject of your email.

▷ Credit: Your report should list all the contributors.

▷ Bugs: Print-outs of your source code are *not* required in your report, unless you have bugs. Help us give you partial marks.

▷ Deadline: Assignments have to be handed in at the beginning of the next exercise session.

Exercise 1 (20 points) Biological Invasion

We are interested in a particular gene χ of a species, which is randomly dispersing in a bounded one dimensional domain Ω. The gene can be present either in the dominant form χ_1 or in the recessive form χ_2. Let $p(x,t)$ be the density of the dominant gene χ_1 present in the population, then Fisher's equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + kp\,(1-p) \tag{1}$$

describes the evolution of p in the population as time progresses.

Part 1 (10 points)

Implement a numerical algorithm to simulate the evolution of the dominant gene with the fisher equation using the explicit euler integration scheme and second-order finite differences in space with dirichlet boundary conditions.

Part 2 (10 points)

Let $\Omega = [0,1]$ denote the space domain. The initial condition is given as

$$p(x,0) = \begin{cases} 1, & \forall x \in [0,1/4] \\ 0, & \forall x \in (1/4,1] \end{cases} \tag{2}$$

The dirichlet boundary conditions are given as

$$p(0,t) = 1\ \forall t \geq 0, \quad p(1,t) = 0\ \forall t \geq 0 \tag{3}$$

Apply your implementation of the first part with the given initial and boundary conditions. Set the diffusion constant $D = 1/160^2$ and reaction constant $k = 1$. Integrate the differential equation from $t = 0$ to $t = 100$ with the time step $\Delta t = \Delta x^2/(20D)$ and plot the density $p(x,t)$ at fixed time intervals $t = [0, 20, 40, 60, 80, 100]$. What do you observe?

Exercise 2 (20 points) von Neumann's Stability Analysis

The problem of shallow wave dispersion involves the following partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \ .$$

We work in a periodic domain $x \in [0, 2\pi]$ and for $t \in [0, \infty]$. This PDE is linear, and we use a uniform grid (constant Δx); we can therefore employ von Neumann analysis to guide our choice of a time integration scheme.

Part 1 (5 points, paper and pencil)

Discretize the spatial differential operator ($\partial^3/\partial x^3$) with centered second-order finite differences.

Part 2 (10 points, paper and pencil)

If we use this spatial discretization and assume spatially periodic solutions, we can write

$$u(x,t) = \sum_{k=0}^{N-1} u_k(x,t) = \sum_{k=0}^{N-1} a_k(t) e^{ikx}$$

where $N = 2\pi/\Delta x$. What is the behavior of $a_k(t)$, i.e. identify the eigenvalues of our scheme.

Part 3 (5 points, paper and pencil)

Discuss the stability of Explicit Euler and Runge-Kutta fourth order for such modes. If applicable, identify the most restrictive mode and derive a stability condition.