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Advanced Computational Science
SS 2009

Serie 3 Ausgabe: March 10, 2009 Abgabe: March 17, 2009

Guidelines

▷ Submission: Submit a written or typed report with your answers and graphics.

▷ Code: Send your source code (and makefile) to the TA of your session. Use "[ACS] Homework1" in the subject of your email.

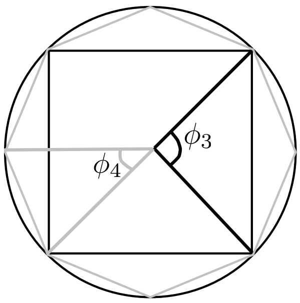
▷ Credit: Your report should list all the contributors.

▷ Bugs: Print-outs of your source code are *not* required in your report, unless you have bugs. Help us give you partial marks.

▷ Deadline: Assignments have to be handed in at the beginning of the next exercise session.

Exercise 1 (15 points) Richardson Extrapolation

Consider the problem of approximating the circumference of a unit circle by an inscribed polygon of n sides.



Let $n_i = \frac{2\pi}{\phi_i}$ denote the number of sides of the polygon after i refinement steps, then the circumference is given by $C_0^{n_i} = 2n_i \sin\left(\frac{\pi}{n_i}\right)$. Starting with the point approximation $C_0^1 = 0$, we proceed with a line $C_0^2 = 4$, then a square approximation C_0^4, etc... as shown in the picture. The sequence of approximations $C_0^{n_1}, C_0^{n_2}, \ldots$ with $n_i = 2^{i-1}$ obtained by successive refinement steps will converge to the exact solution 2π. The approximation and the leading error can be written as

$$C_0^{n_i} = 2\pi + \frac{c_p}{n_i^p} + O\left(\frac{1}{n_i^q}\right), \quad p < q, \tag{1}$$

denoting the exact solution. The idea of Richardson extrapolation is to eliminate the leading error term $\frac{c_p}{n_i^p}$ by a linear combination $C_1^{n_i}$ of two successive approximations $C_0^{n_i}, C_0^{n_{i-1}}$ to achieve a higher convergence rate.

▷ Part 1 (10 Points) Determine the coefficients p, q by expanding $C_0^n = 2n \sin\left(\frac{\pi}{n}\right)$ about $\frac{\pi}{n} = 0$ in a Taylor series.

▷ Part 2 (5 Points) Express $C_k^{n_i}$ as a linear combination of $C_{k-1}^{n_i}$ and $C_{k-1}^{n_{i-1}}$.

▷ Part 2 (10 Points) Implement an algorithm to compute the approximations $C_k^{n_i}$. Compare the relative error $\frac{2\pi - C_0^{n_i}}{2\pi}$ for $i = 1, 2, \ldots, 10$ with no extrapolation used and $\frac{2\pi - C_k^{n_i}}{2\pi}$ for $i = 5$ and $k = 0, \ldots, 4$.

Exercise 2 (15 points) Romberg Integration

Let us use the idea of Richardson extrapolation for integration. Let T_0^n denote the approximation obtained with the composite trapezoidal rule with n equally spaced intervals. Then the error is given by

$$T_0^n = I + c_2 \frac{1}{n^2} + O\left(\frac{1}{n^4}\right) \tag{2}$$

with I denoting the exact integration value.

▷ Part 1 (5 Points) Compute T_1^2 by applying richardson extrapolation to the first two trapezoidal approximations T_0^1 and T_0^2. Show that the resulting equation corresponds to the simpson quadrature rule.

▷ Part 2 (5 Points) Derive the Richardson extrapolation formula for T_k^{2n} given the approximations T_{k-1}^{2n} and T_{k-1}^n.

▷ Part 3 (10 Points) Implement the Romberg integration algorithm using the extrapolation formula you derived in the last part to compute the sequence $T_k^{2^k}$ for $k = 0, 1, \ldots, N$.

▷ Part 4 (5 Points) Apply your implementation of the romberg integration scheme to approximate the integral $\int_0^1 \frac{1}{x+1}\, dx$. Compare the relative error to the composite trapezoidal rule for $k = 0, 1, \ldots, N = 16$.

Exercise 3 (20 points) Fredholm integral equations

A common mathematical problem in physics is that of solving the Fredholm integral equation

$$f(x) = \phi(x) - \int_a^b K(x,t)\phi(t)dt\ .$$

where the functions $f(x)$ and $K(x,t)$ are given and the problem is to obtain $\phi(x)$. This problem appears in areas ranging from fluid mechanics (flow past airfoils), to magnetism (scattering of waves by an obstacle).

▷ Part 1 (5 Points) Describe a numerical method to solve this equation

▷ Part 2 (15 Points) Solve this equation numerically if

$$f(x) = \pi x^2 \tag{3}$$
$$K(x,t) = 3(0.5\sin(3x) - tx^2) \tag{4}$$

and the domain $[a, b]$ is $[0, \pi]$.