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On the generation of vorticity at a free surface

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The mechanism for the generation of vorticity at a viscous free surface is described. This is a free-surface analogue of Lighthill's strategy for determining the vorticity flux at solid boundaries. In this method the zero-shear-stress and pressure boundary conditions are transformed into a boundary integral formulation suitable for the velocity–vorticity description of the flow. A vortex sheet along the free surface is determined by the pressure boundary condition, while the condition of zero shear stress determines the vorticity at the surface. In general, vorticity is generated at free surfaces whenever there is flow past regions of surface curvature. It is shown that vorticity is conserved in free-surface viscous flows. Vorticity which flows out of the fluid across the free surface is gained by the vortex sheet; the integral of vorticity over the entire fluid region plus the integral of 'surface vorticity' over the free surface remains constant. The implications of the present strategy as an algorithm for numerical calculations are discussed.

1. Introduction

Often the term *free surface* is used to refer loosely to any gas–liquid interface. In this paper we define free-surface flows to be idealized gas–liquid flows in which the dynamics of the gas phase is neglected by setting the gas density and viscosity to zero. While we are mostly concerned with free-surface flows in this paper we sometimes discuss the connection between these idealized flows and similar real flows with gas–liquid interfaces.

In free surface flows there are many situations where vorticity enters a flow in the form of a shear layer. This occurs at regions of high surface curvature and superficially resembles separation of a boundary layer at a solid boundary corner, but in the free-surface flow there is very little boundary layer vorticity upstream of the corner and the vorticity which enters the flow is entirely created at the corner. Rood (1994) has associated the flux of vorticity into the flow with the deceleration of a layer of fluid near the surface. These effects are quite clearly seen in spilling breaker flows studied by Duncan *et al.* (1994), Lin & Rockwell (1995) and Dabiri & Gharib (1997).

In this paper we propose a description of free-surface viscous flows in a vortex dynamics formulation. In the vortex dynamics approach to fluid dynamics the emphasis is on the vorticity field which is treated as the primary variable; the velocity is expressed as a functional of the vorticity through the Biot–Savart integral. The pressure no longer appears in the formulation. However, since pressure appears in the free-surface boundary condition, a suitable procedure is required to convert this to a boundary condition on the vorticity.

When vortex dynamics methods are used for viscous flows with solid boundaries a similar problem arises. A vorticity boundary condition may be determined by following Lighthill's (1963) discussion of the problem. Lighthill noted that the velocity field induced by the vorticity in the fluid will not in general satisfy the no-slip boundary condition. The spurious slip velocity may be viewed as a vortex sheet on the surface of the body. In order to enforce the no-slip boundary condition the vortex sheet is distributed diffusively into the flow, transferring the vortex sheet to an equivalent thin viscous vortex layer by means of a vorticity flux. The no-slip condition therefore determines the vorticity flux, which is the strength of the spurious vortex sheet divided by the time increment. The physical character of Lighthill's method has led to its direct formulation and implementation by Kinney and his co-workers (Kinney & Paolino 1974; Kinney & Cielak 1977) in the context of finite difference schemes, and by Koumoutsakos, Leonard & Pepin (1994) to enforce the no-slip boundary condition in the context of vortex methods. Their method has produced benchmark quality simulations of some unsteady flows (Koumoutsakos & Leonard 1995).

For free-surface flows a vortex sheet is employed in order to adjust the irrotational part of the flow. Unlike the case of a solid wall this vortex sheet is part of the vorticity field of the flow and is used to determine the velocity field. The strength of the vortex sheet is determined by enforcing the boundary conditions resulting from a force balance at the free surface. This gives two conditions, which are the subject of this paper: one a relationship between the vorticity flux and the surface acceleration; the other a relationship between the vorticity at the surface and the curvature of the surface. The resulting strategy can be easily adapted to a numerical scheme and can lead to improved numerical methods for the simulation of viscous free-surface flows.

As a conceptually attractive by-product of this study we find that vorticity is conserved if one considers the vortex sheet at the free surface to contain 'surface vorticity'. We prove in §4 that vorticity which fluxes out of the fluid, and appears to be lost, is really gained by the vortex sheet. As an example of the significance of this, consider the approach of a vortex ring at a shallow angle to a free surface. It has been observed (Bernal & Kwon 1989; Gharib 1994) for an air–water interface that the vortex disconnects from itself as it approaches the surface and reconnects to the surface in a U-shaped structure with surface dimples at the vortex ends. There is a clear loss of vorticity from the water and an acceleration of the surface in the direction of motion of the ring as discussed by Rood (1994). In interpreting this from a vortex dynamics point of view we have to distinguish between the real-fluid experiment described and a hypothetical (or numerical) experiment with a free surface. In the free-surface case the missing vorticity is found in a vortex sheet in the surface which connects the vortex ends. We show in the Appendix that vorticity is conserved for two real viscous fluids separated by an interface. This means that vorticity lost from the water passes into the air and would be expected to be found in a fairly thin boundary layer dragged along by the accelerating water surface, similar to the vortex sheet in the free-surface case. We can imagine a limit in which as the viscosity and density of the gas are made smaller and smaller this vortex layer on the gas side of the interface contracts to a vortex sheet.

In §2 we present the governing equations and boundary conditions adapted for a vortex dynamics formulation. In §3 we describe a fractional step strategy for the enforcement of the boundary conditions at a free surface. The conservation of the vorticity field in two-dimensional free-surface flows is shown in §4. Conservation of vorticity in a general three-dimensional context is treated in the Appendix.

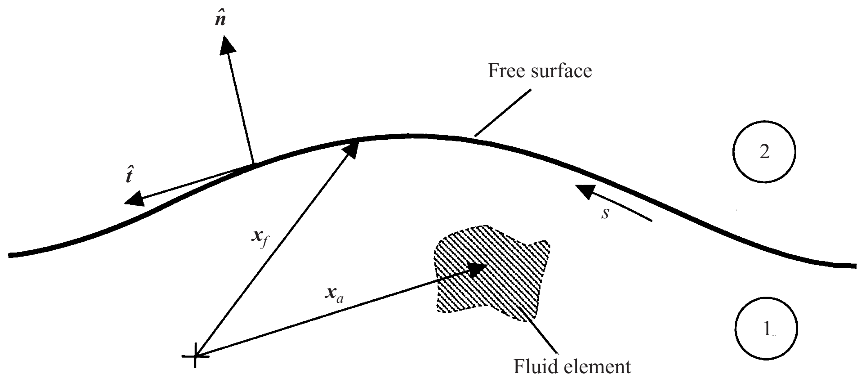

FIGURE 1. Definition sketch.

2. Mathematical formulation

In order to introduce the vorticity generation mechanism we consider, without loss of generality, two-dimensional flow of a Newtonian fluid with a free surface (figure 1). We consider the stresses in *fluid 2* as negligible and when not otherwise stated the flow quantities refer to *fluid 1*.

2.1. *Governing equations*

Two-dimensional incompressible viscous flow may be described by the vorticity transport equation

$$\frac{d\omega}{dt} = \nu\nabla^2\omega \tag{1}$$

with the Lagrangian derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla, \tag{2}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity, $\boldsymbol{\omega} = \omega\hat{\boldsymbol{k}} = \nabla\times\boldsymbol{u}$ the vorticity and ν denotes the kinematic viscosity. The flow field evolves by following the trajectories of the vorticity-carrying fluid elements $\boldsymbol{x}_a$ and the free-surface points $\boldsymbol{x}_f$ based on the following equation:

$$\frac{d\boldsymbol{x}_p}{dt} = \boldsymbol{u}(\boldsymbol{x}_p), \tag{3}$$

where $\boldsymbol{x}_p$ denotes $\boldsymbol{x}_a$ or $\boldsymbol{x}_f$.

2.2. *Boundary conditions*

The boundary conditions at the free surface are determined by a force balance calculation. For a Newtonian fluid the stress tensor is expressed as

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}, \tag{4}$$

where $\boldsymbol{D}$ is the symmetric part of the velocity gradient tensor. The local normal and tangential components of the surface traction force are expressed as $\hat{\boldsymbol{n}}\cdot\boldsymbol{T}\cdot\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}\cdot\boldsymbol{T}\cdot\hat{\boldsymbol{t}}$ respectively. Balancing these two force components results in the following two boundary conditions at a free surface.

(i) Zero shear stress. Assuming negligible surface tension gradients, balancing the tangential forces at the free surface results in

$$\hat{\boldsymbol{t}}\cdot\boldsymbol{D}\cdot\hat{\boldsymbol{n}} = 0. \tag{5}$$

This may be expressed as

$$\hat{\boldsymbol{n}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{t}} + \hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0. \tag{6}$$

For the purposes of our velocity–vorticity formulation we wish to relate this boundary condition to the vorticity field and to the velocity components at the free surface.

For a two-dimensional flow, by the definition of vorticity in a local coordinate system, we have

$$\omega = \hat{\boldsymbol{n}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{t}} - \hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}}. \tag{7}$$

Using (6) we may rewrite (7) as

$$\omega = -2\hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}}. \tag{8}$$

By some further manipulation the free-surface vorticity may be expressed in terms of the local normal and tangential components of the velocity field:

$$\omega = -2\frac{\partial \boldsymbol{u}}{\partial s} \cdot \hat{\boldsymbol{n}} \tag{9}$$

$$= -2\frac{\partial \boldsymbol{u} \cdot \hat{\boldsymbol{n}}}{\partial s} + 2\boldsymbol{u} \cdot \frac{\partial \hat{\boldsymbol{n}}}{\partial s} \tag{10}$$

$$= -2\frac{\partial \boldsymbol{u} \cdot \hat{\boldsymbol{n}}}{\partial s} + 2\boldsymbol{u} \cdot \hat{\boldsymbol{t}}\ \kappa, \tag{11}$$

where κ is the curvature of the surface, defined by $\kappa = \hat{\boldsymbol{t}} \cdot \partial \hat{\boldsymbol{n}} / \partial s$. For steady flow, where the free surface is stationary, $\boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}$ is zero and the first term on the right of (11) drops out. The steady version of (11) was given by Lugt (1987) and by Longuet-Higgins (1992), the unsteady form by Wu (1995). A three-dimensional version of (8) was derived by Lundgren (1989).

The sense of (11) is that vorticity develops at the surface whenever there is relative flow along a curved interface. This condition prevents a viscous free-surface flow from being irrotational. Enforcing the vorticity field given by the above equation at the free surface is equivalent to enforcing the condition of zero shear stress.

(ii) Pressure boundary condition. This is the condition that the jump in normal traction across the free-surface interface is balanced by the surface tension. It is expressed as

$$\|\hat{\boldsymbol{n}} \cdot \boldsymbol{T} \cdot \hat{\boldsymbol{n}}\| = -T\kappa \tag{12}$$

where T is the surface tension and the vertical lines denote the jump in the quantity. Using (4), this becomes

$$-p_1 + 2\mu \hat{\boldsymbol{n}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}} + p_2 = -T\kappa. \tag{13}$$

Using the continuity equation, expressed in local coordinates, we get

$$\hat{\boldsymbol{n}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}} = -\hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{t}} \tag{14}$$

$$= -\frac{\partial \boldsymbol{u} \cdot \hat{\boldsymbol{t}}}{\partial s} + \boldsymbol{u} \cdot \frac{\partial \hat{\boldsymbol{t}}}{\partial s} \tag{15}$$

$$= -\frac{\partial \boldsymbol{u} \cdot \hat{\boldsymbol{t}}}{\partial s} - \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\ \kappa. \tag{16}$$

Therefore

$$p_1 = p_2 + T\kappa - 2\rho\nu\left(\frac{\partial \boldsymbol{u} \cdot \hat{\boldsymbol{t}}}{\partial s} + \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\ \kappa\right) \tag{17}$$

where p_2 is the constant pressure on the zero-density side of the interface.

Since pressure does not occur in the vorticity equation, the pressure condition must be put in a form which accesses the primary variables. From the momentum equation at the free surface we obtain

$$\hat{\boldsymbol{t}} \cdot \frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial s} + \nu \hat{\boldsymbol{n}} \cdot \nabla \omega - g \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{t}} \tag{18}$$

where g is the gravitational constant; $\hat{\boldsymbol{j}}$ is upward.

For our purposes this equation may be put in a more tractable form by further manipulation. First we observe that

$$\hat{\boldsymbol{t}} \cdot \frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}}{\mathrm{d}t} + \boldsymbol{u}_1 \cdot \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}t} \tag{19}$$

and

$$\boldsymbol{u}_1 \cdot \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}t} = \boldsymbol{u}_1 \cdot \ \hat{\boldsymbol{n}}\, \hat{\boldsymbol{n}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}t}. \tag{20}$$

Then using the fact that the free surface is a material surface we obtain the kinematic identity

$$\hat{\boldsymbol{n}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}t} = \hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}} \tag{21}$$

$$= \frac{\partial \boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}}{\partial s} - \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}\ \kappa. \tag{22}$$

Using this identity we find

$$\frac{\mathrm{d}\boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}}{\mathrm{d}t} = \boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}\ \frac{\partial \boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}}{\partial s} - \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}\ \boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}}\ \kappa - \frac{1}{\rho}\frac{\partial p_1}{\partial s} + \nu \hat{\boldsymbol{n}} \cdot \nabla \omega - g \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{t}}. \tag{23}$$

We emphasize that the material derivative here is taken following a fluid particle on side 1 of the interface.

With p_1 substituted from (17) equation (23) may be regarded as equivalent to the pressure boundary condition. Except for the flux term all the terms on the right-hand side of the equation are quantities defined on the surface and derivatives of these along the surface. We prefer to think of the role of the vorticity flux in this equation as a term which modifies the surface acceleration, rather than consider that the equation determines the flux.

Using a strategy analogous to Lighthill's for a solid wall, we propose a fractional step algorithm that enforces the pressure boundary condition in a vorticity–velocity framework. This strategy allows us to gain insight into the development and generation of vorticity at a viscous free surface and can be used as a building tool for a numerical method.

3. A fractional step algorithm

In order to show that the free-surface boundary conditions are satisfied in a velocity–vorticity formulation we consider the evolution of the flow field during a single time step. In a manner similar to Lighthill's approach for a solid boundary, a vortex sheet is employed to enforce the boundary conditions. The vortex sheet becomes part of the vorticity field of the flow. The difference between the solid wall and the free surface is the role of the surface vortex sheet in adjusting the velocity field of the flow. In the case of the solid wall the vortex sheet is eliminated from the boundary (so that the no-slip boundary condition is enforced) and enters the flow diffusively, resulting in the flux of vorticity into the flow field. In the case of a free surface the vortex sheet remains at the surface to enforce the pressure boundary condition and constitutes a part of the vorticity field of the flow. The task is to determine the strength of the vortex sheet at the free surface so as to satisfy the boundary conditions.

For the purpose of describing this process we assume that the velocity and the vorticity field are known at time t^n throughout the flow field and at the free surface and we wish to obtain the flow field at time $t^{n+1}(\equiv t^n + \delta t)$.

Step 1. Given the velocity and vorticity at time t^n we update the positions of the vorticity-carrying elements and the surface markers by solving $\mathrm{d}\boldsymbol{x}_p/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x}_p, t)$:

$$\boldsymbol{x}_p^{n+1} = \boldsymbol{x}_p^n + \delta t \boldsymbol{u}^n(\boldsymbol{x}_p^n). \tag{24}$$

We update the vorticity field by solving

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \nu \nabla^2 \omega \tag{25}$$

with initial condition $\omega = \omega^n$ at $t = t^n$ and boundary condition $\omega = \omega^n(\boldsymbol{x}_f)$ at $\boldsymbol{x} = \boldsymbol{x}_f$. The solution to this equation, which we denote by $\omega^{n+1/2}$, is still incomplete. It does not satisfy the correct vorticity boundary condition at the end of the time step and must be corrected in Step 2. The boundary condition which we have imposed ensures (rather arbitrarily) that the vorticity on the boundary is purely convected. The correction which is needed will be a vortical layer along the free surface with vorticity of order δt and with thickness of order $(\delta t)^{1/2}$. We reason that the additional velocity field induced across this layer can be neglected, since its variation is only of order $(\delta t)^{3/2}$.

For an incompressible flow the velocity may be expressed in terms of a stream function ψ by

$$\boldsymbol{u} = -\hat{\boldsymbol{k}} \times \nabla \psi \tag{26}$$

and the vorticity itself is related to ψ by

$$\omega \equiv \hat{\boldsymbol{k}} \cdot \nabla \times \boldsymbol{u} = -\nabla^2 \psi. \tag{27}$$

We use the convention that $\hat{\boldsymbol{n}}$ is always outward from the fluid, $\hat{\boldsymbol{t}}$ is the direction of integration along the surface, and $\hat{\boldsymbol{k}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{t}}$ is a unit vector out of the page. The solution of this equation gives

$$\psi = \psi_\omega + \psi_\gamma \tag{28}$$

where

$$\psi_\omega(\boldsymbol{x}) = -\frac{1}{2\pi} \int_{\text{fluid}} \omega(\boldsymbol{x}_a, t) \ln |\boldsymbol{x} - \boldsymbol{x}_a| \mathrm{d}\boldsymbol{x}_a \tag{29}$$

and ψ_γ represents an irrotational flow selected to satisfy boundary conditions. It is consistent with vortex dynamics to take this irrotational part as the flow induced by a vortex sheet along the boundary of the fluid, i.e. by

$$\psi_\gamma(\boldsymbol{x}, t) = -\frac{1}{2\pi} \int_{\text{intfc}} \gamma(\boldsymbol{x}_f(s'), t) \ln |\boldsymbol{x} - \boldsymbol{x}_f(s')| \mathrm{d}s', \tag{30}$$

but it must be shown that this can be done in such a way as to satisfy the boundary conditions. In this formulation the boundary can be either solid or free or a mix of these, but in this paper we are specifically interested in free boundaries which separate an incompressible fluid from a fluid of negligible mass density. The velocity field is obtained by applying (26), giving the Biot–Savart law:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_\omega(\boldsymbol{x}, t) + \boldsymbol{u}_\gamma(\boldsymbol{x}, t) \tag{31}$$

where

$$\boldsymbol{u}_\omega(\boldsymbol{x}) = \frac{1}{2\pi} \int_{\text{fluid}} \boldsymbol{\omega}(\boldsymbol{x}_a, t)\ \hat{\boldsymbol{k}} \times \nabla \ln |\boldsymbol{x} - \boldsymbol{x}_a| \mathrm{d}\boldsymbol{x}_a \tag{32}$$

and

$$\boldsymbol{u}_\gamma(\boldsymbol{x}) = \frac{1}{2\pi} \int_{\text{intfc}} \gamma(\boldsymbol{x}_f(s), t)\ \hat{\boldsymbol{k}} \times \nabla \ln |\boldsymbol{x} - \boldsymbol{x}_f(s)| \mathrm{d}s. \tag{33}$$

The velocity field is also defined by these integrals for points outside the fluid: $\boldsymbol{u}_\omega$ is continuous across the interface, $\boldsymbol{u}_\gamma$ has a jump discontinuity. As the position vector $\boldsymbol{x}$ tends to a point on the interface from inside the fluid, which we will indicate with a subscript 1, we get

$$(\boldsymbol{u}_\gamma \cdot \hat{\boldsymbol{t}})_1 = -\frac{\gamma(s)}{2} - \mathrm{PV}\frac{1}{2\pi} \int_{\text{intfc}} \gamma(s', t)\ \hat{\boldsymbol{n}} \cdot \nabla \ln |\boldsymbol{x}_f(s) - \boldsymbol{x}_f(s')| \mathrm{d}s' \tag{34}$$

while as the point is approached from the outside, indicated by 2,

$$(\boldsymbol{u}_\gamma \cdot \hat{\boldsymbol{t}})_2 = +\frac{\gamma(s)}{2} - \mathrm{PV}\frac{1}{2\pi} \int_{\text{intfc}} \gamma(s', t)\ \hat{n} \cdot \nabla \ln |\boldsymbol{x}_f(s) - \boldsymbol{x}_f(s')| \mathrm{d}s'. \tag{35}$$

Here PV indicates the principal value of these singular integrals. By subtracting these equations it is clear that the vortex sheet strength is the jump in tangential velocity across the interface; since $\boldsymbol{u}_\omega \cdot \hat{\boldsymbol{t}}$ is continuous we have

$$\gamma = \boldsymbol{u}_2 \cdot \hat{\boldsymbol{t}} - \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}. \tag{36}$$

By (31) and (34) the tangential component of the surface velocity is

$$-\frac{\gamma(s)}{2} - \mathrm{PV}\frac{1}{2\pi} \int_{\text{intfc}} \gamma(s', t)\ \hat{\boldsymbol{n}} \cdot \nabla \ln |\boldsymbol{x}_f(s) - \boldsymbol{x}_f(s')| \mathrm{d}s' = \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}} - (\boldsymbol{u}_\omega \cdot \hat{\boldsymbol{t}})_1. \tag{37}$$

Equation (37) is a Fredholm integral equation of the second kind the solution of which determines the strength (γ) of the free-surface vortex sheet when the right-hand side is given. In the case of multiply connected domains the equation needs to be supplemented with m constraints for the strength of the vortex sheet, where $m+1$ is the multiplicity of the domain (Prager 1928). For example in the case of a free surface extending to infinity no additional constraint needs to be imposed as the problem involves integration over a singly connected domain. However, in the case of a bubble, an additional constraint, such as the conservation of total circulation in the domain, needs to be imposed in order to obtain a unique solution.

The right-hand side of the equation may be determined from the quantities which have been updated. In particular $\boldsymbol{u}_\omega$ can be computed via the Biot–Savart integral (32) from the known vorticity field $\omega^{n+1/2}$ with order-δt accuracy. The tangential component of the velocity of the free surface can be computed using (23) in the form

$$(\boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}})^{n+1} = (\boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}})^n + \delta t Q^n \left(\boldsymbol{u}_1, \hat{\boldsymbol{n}}, \hat{\boldsymbol{t}}, \nu \frac{\partial \omega}{\partial n}, p_1 \right) \tag{38}$$

where Q^n signifies the right-hand side of (23) evaluated at time t^n. The pressure boundary condition enters the formulation of the problem at this stage. Upon solving

(37) the strength of the vortex sheet is determined such that the pressure boundary condition is satisfied, justifying the previous assertion.

Note that the present method of enforcing the pressure boundary condition is equivalent to previous irrotational formulations (Lundgren & Mansour 1988, 1991) which employ a velocity potential.

At the end of this step the points of the free surface, the velocity field and the strength of the vortex sheet have been updated (${\boldsymbol{x}_p}^{n+1}$, $\boldsymbol{u}^{n+1}$ and γ^{n+1}). The vorticity field ($\omega^{n+1/2}$) still needs to be corrected near the free surface.

Step 2. At this step we consider generation of vorticity at the free surface. Having determined the strength of the vortex sheet from Step 1 we can compute the normal and tangential components of the velocity field at the free surface in order to determine the free-surface vorticity and enforce the zero-shear-stress boundary condition.

Using (28)–(30) we can compute an updated value of the stream function on the surface and from this compute $\boldsymbol{u}_1 \cdot \hat{\boldsymbol{n}} = \partial\psi/\partial s$. Since the surface shape and $\boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}$ have already been updated we have all the ingredients necessary to compute an updated value of ω_1 from (7). The next step in this process is to solve the vorticity transport equation for the vorticity field using ω_1 as boundary condition. For the final partial step we need to solve the heat equation,

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega, \tag{39}$$

with initial condition $\omega = 0$ at $t = t^n$, and with the boundary condition

$$\omega(\boldsymbol{x}_f) = (\omega_1^{n+1} - \omega_1^n)(t - t^n)/\delta t \tag{40}$$

assuming a linear time variation of the surface vorticity between the two time levels. The solution of this partial step is to be added to $\omega^{n+1/2}$ thus yielding the completely updated vorticity field ω^{n+1}.

An analytical solution for this diffusion equation can be obtained using the method of heat potentials (Friedman 1966). For a two-dimensional flow the solution to the above equation may be expressed in terms of double-layer heat potentials as

$$\omega(\boldsymbol{x}, t + \delta t) = \int_t^{t+\delta t} \int_{\text{intfc}} \frac{\partial G}{\partial n'}(\boldsymbol{x} - \boldsymbol{x}_f(s'), t - t') \mu(s', t')\, \mathrm{d}s'\, \mathrm{d}t' \tag{41}$$

where G is the fundamental solution of the heat equation and the function $\mu(s, t)$ is determined by the solution of the following second-order Fredholm integral equation:

$$-\tfrac{1}{2}\mu(s,t) + \int_t^{t+\delta t} \int_{\text{intfc}} \mu(s', t) \frac{\partial G}{\partial n'}(\boldsymbol{x}_f(s) - \boldsymbol{x}_f(s'), t - t')\, \mathrm{d}s'\, \mathrm{d}t' = \omega(\boldsymbol{x}_f(s), t). \tag{42}$$

Following Greengard & Strain (1990) and Koumoutsakos *et al.* (1994) we can obtain asymptotic formulae for the above integrals. Similar formulae could help in the development of a numerical method based on the proposed algorithm.

This update strategy was posed without requiring any particular numerical methods for the computational steps. We have particular methods in mind, however, for using this strategy for future numerical work. We will use a boundary integral method similar to that used by Lundgren & Mansour (1988, 1991) for the surface computations. That work was for irrotational inviscid flow. Instead of the pressure boundary condition in the form of (23), an unsteady Bernoulli equation was used to access the pressure.

For the vortical part of the flow we propose to use the point vortex method employed by Koumoutsakos *et al.* (1994) and Koumoutsakos & Leonard (1995) for viscous flow problems with solid boundaries. In these problems the Lighthill strategy provides a vorticity flux boundary condition for the second step in the vorticity update, a Neuman condition. In the proposed free-surface strategy a Dirichlet condition is required for the second vorticity step. This modification can be accomplished by using double-layer heat potentials (as suggested above) where single-layer potentials were used in the solid boundary work.

4. Conservation of vorticity

We will show that vorticity is conserved in two-dimensional free-surface problems: vorticity which flows through the free surface does not disappear but resides in the vortex sheet along the surface. (This is shown for general three-dimensional flows in the Appendix.)

In the interior of the fluid it is easy to show from Helmholtz's equation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{A_1} \omega \mathrm{d}A = \int_{S_1} \nu \frac{\partial \omega}{\partial n} \mathrm{d}s, \tag{43}$$

where A_1 is a material 'volume' and S_1 its 'surface', n is outward from the region and $-\nu \partial\omega/\partial n$ is the vorticity flux in the outward direction. This says that the vorticity in A_1 increases because of viscous vorticity flux into the region; there are no vorticity sources in the interior of the fluid.

Everything we need to know about the velocity on side 2 is contained in (31)–(33). We will only use the fact that, because the velocity on side 2 is irrotational, there must be a velocity potential ($\boldsymbol{u}_2 = \nabla \Phi$). We use $\mathrm{d}/\mathrm{d}t$ to mean the material derivative along side 1, and note that $\boldsymbol{u}_2 - \boldsymbol{u}_1 = \gamma \hat{\boldsymbol{t}}$; then by some simple manipulations

$$\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \frac{\partial \boldsymbol{u}_2}{\partial t} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_2 \tag{44}$$

$$= \nabla\left(\frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}\boldsymbol{u}_2 \cdot \boldsymbol{u}_2\right) - \gamma \hat{\boldsymbol{t}} \cdot \nabla \gamma \hat{\boldsymbol{t}} - \gamma \hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u}_1. \tag{45}$$

Then

$$\hat{\boldsymbol{t}} \cdot \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \frac{\partial}{\partial s}\left(\frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 - \tfrac{1}{2}\gamma^2\right) - \gamma \hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}. \tag{46}$$

The last term in this equation is the strain rate of a surface element and may be expressed as

$$\hat{\boldsymbol{t}} \cdot \nabla \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}} = \frac{1}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}s, \tag{47}$$

where $\mathrm{d}s$ is a material line element on side 1. Subtracting (18) from (46) then gives

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} + \frac{\gamma}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}s = \frac{\partial}{\partial s}\left(\frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 - \tfrac{1}{2}\gamma^2 + \frac{p_1}{\rho} + gy\right) - \nu \hat{\boldsymbol{n}} \cdot \nabla \omega. \tag{48}$$

This may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma \mathrm{d}s = -\nu \frac{\partial \omega}{\partial n}\mathrm{d}s - \frac{\partial \Phi}{\partial s}\mathrm{d}s \tag{49}$$

with Φ given by

$$\Phi = -\left[\frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 - \tfrac{1}{2}\gamma^2 + gy\right] - \frac{p_1}{\rho}. \tag{50}$$

If we integrate (49) over a material segment along the interface we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \gamma \mathrm{d}s = -\int_a^b \nu \frac{\partial \omega}{\partial n}\mathrm{d}s - \int_a^b \frac{\partial \Phi}{\partial s}\mathrm{d}s. \tag{51}$$

From this form we see that Φ should be interpreted as a surface-vorticity flux. Since γ is a density (circulation density or surface-vorticity density) the last term in (51), which may be written $\Phi_a - \Phi_b$, is the flux of surface vorticity into the interval at a minus the flux out at b, while the first term on the right is the flux of vorticity into the interval through the surface.

If the interval is extended over the entire interface, by extending it to infinity for an 'ocean', or continuing b around to a for a closed interface, like a bubble, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{intfc}} \gamma \mathrm{d}s = -\int_{\mathrm{intfc}} \nu \frac{\partial \omega}{\partial n}\mathrm{d}s. \tag{52}$$

Now letting A_1 in (43) be the entire fluid we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{fluid}} \omega \mathrm{d}A = \int_{\mathrm{intfc}} \nu \frac{\partial \omega}{\partial n}\mathrm{d}s. \tag{53}$$

Adding (53) and (52) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{fluid}} \boldsymbol{\omega} \mathrm{d}A + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{intfc}} \gamma \mathrm{d}s = 0. \tag{54}$$

It is in this sense that vorticity is conserved.

We began this approach as an attempt to obtain an evolution equation for γ which would eliminate the need to solve an integral equation, (37), to update γ. Equation (48) or (49) might appear to play such a role, but the occurrence of the velocity potential ϕ_2 in the equation makes it unuseable for this purpose. Since ϕ_2 could be expressed by an integration over the surface involving γ, the time derivative of ϕ_2 would involve a surface integral of $\mathrm{d}\gamma/\mathrm{d}t$ and therefore an integral equation for $\mathrm{d}\gamma/\mathrm{d}t$ would result, defeating the purpose.

4.1. *Pedley's problem: vorticity outside a swirling cylindrical bubble*

A problem solved by Pedley (1967) as part of a study on the stability of swirling toroidal bubbles gives an example which illustrates some concepts discussed here. One can describe the flow as a potential vortex of circulation Γ swirling around a bubble cavity of radius R. The flow is induced by a vortex sheet of strength $\gamma_0 = \Gamma/2\pi R$ at the bubble interface. At some initial time one turns on the viscosity and vorticity begins to leak from the vortex sheet into the fluid. The circulation at infinity remains constant and therefore the strength of the vortex sheet must decrease with time.

We pose this problem in the form described in § 2. Since the flow is axially symmetric the vorticity satisfies

$$\frac{\partial \omega}{\partial t} = \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right). \tag{55}$$

The vorticity boundary condition (11) is

$$\omega_1 = -2V_1/R, \tag{56}$$

where $V_1 = \boldsymbol{u}_1 \cdot \hat{\boldsymbol{t}}$ is the tangential component of the velocity at the interface (with the tangent convention used earlier V_1 is negative for positive swirl), and R is the constant radius of curvature of the surface. The pressure boundary condition (23) is

$$\frac{\partial V_1}{\partial t} = -\nu\left(\frac{\partial \omega}{\partial r}\right)_1. \tag{57}$$

The velocity inside the bubble is zero so $\boldsymbol{u}_2 \cdot \hat{\boldsymbol{t}} = 0$. The strength of the vortex sheet is therefore $\gamma = -V_1$, a positive quantity. The sense of the problem is that since ω_1 is required to be non-zero a layer of positive vorticity must develop in the fluid. The resulting flux of vorticity out of the interface causes γ to decrease with time.

Equations (56) and (57) may be combined into a single boundary condition

$$\frac{\partial \omega_1}{\partial t} = \frac{2\nu}{R}\left(\frac{\partial \omega}{\partial r}\right)_1. \tag{58}$$

Therefore the problem is to solve (55) with this boundary condition and with initial conditions $\omega = 0$ for all $r > R$ and $\omega = 2\gamma_0/R$ for $r = R$. This last condition prevents the trivial solution.

For large $\tau(\equiv \nu t/R^2)$ Pedley gives an approximate solution:

$$\omega = \frac{\pi\gamma_0}{2R\tau}\exp\left(-\frac{r^2}{4R^2\tau}\right). \tag{59}$$

This satisfies (55) exactly, but has a relative error of order τ^{-1} in the boundary condition. For small τ another approximate similarity solution is

$$\omega = \frac{2\gamma_0}{R}\exp(2x + 4\tau)\,\mathrm{erfc}\left(\frac{x}{2\tau^{1/2}} + 2\tau^{1/2}\right) \tag{60}$$

where $x = (r - R)/R$. This solution satisfies the boundary condition exactly but neglects the last term in (55), requiring that τ be small enough that the vortical layer is thin compared to the radius of the bubble.

Further details of the solution are unimportant here. This problem illustrates both conservation of vorticity and generation of vorticity when there is flow along a curved free surface.

5. Conclusion

In this paper we have presented a strategy for solving free-surface viscous flow problems in a vortex dynamics formulation. This strategy centres on determining suitable boundary conditions for the vorticity in analogy with Lighthill's strategy for solid-boundary flows. The two free-surface boundary conditions play distinct roles in determining free-surface viscous flows. We have shown that the pressure boundary condition determines the strength of a vortex sheet at the free surface, which determines the irrotational part of the flow. The pressure force modifies the surface velocity, from which the vortex sheet strength is found by solving an integral equation. The zero-shear-stress boundary condition, on the other hand, determines the value of the vorticity at the surface, providing a Dirichlet condition for the vorticity equation.

We have shown that vorticity is conserved for both two- and three-dimensional free-surface flows, the vortex sheet being considered part of the vorticity field. It follows that vorticity which might appear to be lost by flux across the free surface now resides in the vortex sheet. It is shown in the Appendix that vorticity is conserved for two viscous fluids in contact across an interface. It is physically clear that in the limit as the density and viscosity of one of the fluids becomes small, the vorticity transmitted to that fluid would be confined to a thin surface layer, a vortex sheet in the limit. Kelvin has shown (Lamb 1932, art. 145) that vortex lines cannot begin nor end in the interior of a fluid. This is also true for two viscous fluids in contact, since the proof only requires continuity of the velocity field. The vortex dynamics formulation suggests that vortex lines do not end at free surfaces (where the velocity is discontinuous) but can abruptly bend and continue in the surface to complete a closed circuit. This is a physically reasonable result in the light of the limit process described, but cannot be proved directly in terms of vortex tubes, since a tube in the free surface would have infinite vorticity in zero cross-section.

Appendix. Vorticity conservation in three-dimensional flows

A.1. *Real viscous fluids separated by an interface*

In three-dimensional flows there is a simple kinematic result which was shown by Truesdell (1953). For smooth vorticity fields, which tend to zero fast enough at infinity, vorticity is conserved in the form

$$\int_{\text{fluid}} \boldsymbol{\omega} \mathrm{d}V = 0. \tag{A 1}$$

This means, for instance, that the average vorticity in a vortex ring is zero: vorticity on one part of the ring is cancelled by vorticity in the opposite direction on another part of the ring. Equation (A 1) can be proved by using the identity

$$\boldsymbol{\omega} = \nabla \cdot (\boldsymbol{\omega} \boldsymbol{r}), \tag{A 2}$$

(which requires only $\nabla \cdot \boldsymbol{\omega} = 0$) where $\boldsymbol{r}$ is the position vector. Integrating this over a finite volume V and using the divergence theorem gives

$$\int_V \boldsymbol{\omega} \mathrm{d}V = \int_S \hat{\boldsymbol{n}} \cdot \boldsymbol{\omega} \boldsymbol{r} \mathrm{d}S. \tag{A 3}$$

Equation (A 1) follows upon letting V be the whole space. It is easy to see that (A 1) is still true if there is a stationary solid body included in a viscous fluid since $\boldsymbol{\omega} \cdot \hat{\boldsymbol{n}} = 0$ at no-slip boundaries.

Consider now a case where two real fluids with different finite viscosities are separated by an interface. The vorticity is discontinuous so (A 3) is only valid for volumes on either side of the interface. Since the velocity is continuous in viscous fluids it follows that $\boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}$ is continuous across the interface and therefore, by cancellation of the two surface integrals, we see that (A 1) is still true in this case. However, if one of the fluids is non-viscous, as for free-surface problems, one can no longer assume continuity of the velocity and (A 1) cannot be proved.

A.2. *Free-surface flows*

For three-dimensional free-surface flows we will prove that

$$\int_{\text{fluid}} \boldsymbol{\omega} \mathrm{d}V + \int_{\text{intfc}} \boldsymbol{\gamma} \mathrm{d}S \equiv 0, \tag{A 4}$$

where the vortex sheet strength is defined by

$$\boldsymbol{\gamma} = \hat{\boldsymbol{n}} \times (\boldsymbol{u}_2 - \boldsymbol{u}_1), \tag{A 5}$$

this being the circulation per unit length around a surface element in the plane of the velocity jump. Only the Biot–Savart law is required for the proof; the fluids do not need to satisfy the Navier–Stokes equations. Consider the domain to consist of a vortical incompressible fluid on side 1 separated from an irrotational fluid on side 2 by an interface which extends to infinity. The boundary of the vortical fluid is the interface plus a surface S_1 which will be taken to infinity. The irrotational part is bounded by the interface and a distant surface S_2. Since $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ we have, by Gauss's theorem,

$$\int_{\text{fluid}} \omega \mathrm{d}V = \int_{\text{intfc}} \hat{\boldsymbol{n}} \times \boldsymbol{u}_1 \, \mathrm{d}S + \int_{S_1} \hat{\boldsymbol{n}} \times \boldsymbol{u} \mathrm{d}S. \tag{A 6}$$

Since vorticity is zero on side 2 we can also write

$$0 = -\int_{\text{intfc}} \hat{\boldsymbol{n}} \times \boldsymbol{u}_2 \, \mathrm{d}S + \int_{S_2} \hat{\boldsymbol{n}} \times \boldsymbol{u} \, \mathrm{d}S, \tag{A 7}$$

where the normal on the interface is directed toward side 2. Adding these equations, and defining a temporary quantity $\boldsymbol{\Omega}$, we find

$$\boldsymbol{\Omega} \equiv \int_{\text{fluid}} \omega \mathrm{d}V + \int_{\text{intfc}} \gamma \mathrm{d}S = \int_{S_1 \cup S_2} \hat{\boldsymbol{n}} \times \boldsymbol{u} \mathrm{d}S. \tag{A 8}$$

The velocity field induced by the vorticity field of the fluid and the vortex sheet is given by the three-dimensional Biot–Savart law

$$\boldsymbol{u} = -\int_{\text{fluid}} \nabla g(\boldsymbol{r}, \boldsymbol{r}') \times \boldsymbol{\omega}' \mathrm{d}V' - \int_{\text{intfc}} \nabla g(\boldsymbol{r}, \boldsymbol{r}') \times \boldsymbol{\gamma}' \mathrm{d}S', \tag{A 9}$$

where

$$g(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|}, \tag{A 10}$$

assuming that the velocity is zero at infinity. If the vorticity and vortex sheet are sufficiently compact, say zero outside of some region, then

$$\boldsymbol{u} = \frac{1}{4\pi} \frac{\boldsymbol{r}}{r^3} \times \boldsymbol{\Omega}, \tag{A 11}$$

asymptotically for large r. Now consider this asymptotic result in the integral on the right-hand side of (A 8). If the distant surface is a sphere centred at the origin the integral is zero since then $\boldsymbol{r}$ is in the same direction as $\hat{\boldsymbol{n}}$. But this result is independent of the shape of the distant surface. Since vorticity is zero in the distant region one can apply Gauss's theorem in the region between two distant surfaces, obtaining

$$\int_{\text{Surf 1}} \hat{\boldsymbol{n}} \times \boldsymbol{u} \, \mathrm{d}S = \int_{\text{Surf 2}} \hat{\boldsymbol{n}} \times \boldsymbol{u} \, \mathrm{d}S. \tag{A 12}$$

Since the integral is zero for a sphere it is zero in general. We have thus proved that $\boldsymbol{\Omega} = 0$. Therefore (A 4) is true.

The physical connection with the case of two viscous fluids is clear. If one considers the limit as the viscosity of the second fluid becomes small, the vorticity field in that fluid will reduce to a thin vortical layer along the interface and (A 1) will reduce to (A 4).

We can show the more limited result that $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = 0$ using the method employed for two-dimensional free-surface flows in § 4. This is analytically intensive but very instructive. Using the definition (A 5) the material derivative of $\boldsymbol{\gamma}$ following the fluid on side 1 of the free surface is

$$\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t} = \hat{\boldsymbol{n}} \times \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_2 - \boldsymbol{u}_1) + \frac{\mathrm{d}\hat{\boldsymbol{n}}}{\mathrm{d}t} \times (\boldsymbol{u}_2 - \boldsymbol{u}_1). \tag{A 13}$$

Using the kinematic equation $\mathrm{d}\,\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \mathrm{d}\boldsymbol{r}\cdot\nabla\boldsymbol{u}$ for a fluid element normal to the surface, one can show

$$\frac{\mathrm{d}\hat{\boldsymbol{n}}}{\mathrm{d}t} = \hat{\boldsymbol{n}}\cdot\nabla\boldsymbol{u}_1 - \hat{\boldsymbol{n}}\cdot\nabla\boldsymbol{u}_1\cdot\hat{\boldsymbol{n}}\,\hat{\boldsymbol{n}} \tag{A 14}$$

and therefore using this and $(\boldsymbol{u}_2-\boldsymbol{u}_1) = -\hat{\boldsymbol{n}}\times\gamma$ we find

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \hat{\boldsymbol{n}}\times\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_2-\boldsymbol{u}_1) - \hat{\boldsymbol{n}}\,\hat{\boldsymbol{n}}\cdot\nabla\boldsymbol{u}_1\cdot\gamma. \tag{A 15}$$

From the Navier–Stokes equation for the fluid on side 1 one can easily obtain

$$\hat{\boldsymbol{n}}\times\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = -\hat{\boldsymbol{n}}\times\nabla\left(\frac{p_1}{\rho}+gy\right) - \nu\nabla\omega_1\cdot\hat{\boldsymbol{n}} + \nu\hat{\boldsymbol{n}}\cdot\nabla\omega_1. \tag{A 16}$$

There is no Navier–Stokes equation on side 2 since the density is zero, but an irrotational velocity field is defined there by the Biot–Savart integral. Following an analysis similar to the two-dimensional case, we have

$$\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \frac{\partial\boldsymbol{u}_2}{\partial t} + \boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_2 \tag{A 17}$$

$$= \nabla\frac{\partial\phi_2}{\partial t} + (\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u}_2 + \nabla\frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}{2} \tag{A 18}$$

$$= \nabla\left(\frac{\partial\phi_2}{\partial t} + \frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}{2}\right) + (\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u}_1 - (\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla(\boldsymbol{u}_2-\boldsymbol{u}_1) \tag{A 19}$$

and therefore

$$\hat{\boldsymbol{n}}\times\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \hat{\boldsymbol{n}}\times\nabla\left(\frac{\partial\phi_2}{\partial t} + \frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}{2}\right) + (\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla\boldsymbol{u}_1\times\hat{\boldsymbol{n}} + (\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla(\boldsymbol{u}_2-\boldsymbol{u}_1)\times\hat{\boldsymbol{n}}. \tag{A 20}$$

The last term may be written

$$(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla(\boldsymbol{u}_2-\boldsymbol{u}_1)\times\hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}\times\nabla\frac{(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot(\boldsymbol{u}_2-\boldsymbol{u}_1)}{2} - (\boldsymbol{u}_2-\boldsymbol{u}_1)\,\omega_1\cdot\hat{\boldsymbol{n}} \tag{A 21}$$

by using the vector identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\boldsymbol{u}^2/2 - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$. The second term on the right of (A 20) requires more manipulation to get it into the desired form. First write

$$(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla\boldsymbol{u}_1\times\hat{\boldsymbol{n}} = \gamma\cdot(\hat{\boldsymbol{n}}\times\nabla\boldsymbol{u}_1\times\hat{\boldsymbol{n}}). \tag{A 22}$$

Then by expanding the dyadic $\nabla\boldsymbol{u}_1$ in a local coordinate system with orthogonal base vectors $\hat{\boldsymbol{t}}_1$, $\hat{\boldsymbol{t}}_2$, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}}_1\times\hat{\boldsymbol{t}}_2$ we find

$$(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla\boldsymbol{u}_1\times\hat{\boldsymbol{n}} = -\gamma\;\nabla_S\cdot\boldsymbol{u}_1 + \gamma\cdot\nabla_S\boldsymbol{u}_1 - \gamma\cdot\nabla_S\boldsymbol{u}_1\cdot\hat{\boldsymbol{n}}\,\hat{\boldsymbol{n}} + (\boldsymbol{u}_2-\boldsymbol{u}_1)\,\omega_1\cdot\hat{\boldsymbol{n}}. \tag{A 23}$$

The symbol ∇_S is the surface component of ∇: $\nabla_S = \nabla - \hat{\boldsymbol{n}}\,\hat{\boldsymbol{n}}\cdot\nabla$. Substituting (A 21) and (A 23) into (A 20) gives

$$\hat{\boldsymbol{n}}\times\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \hat{\boldsymbol{n}}\times\nabla\left(\frac{\partial\phi_2}{\partial t} + \frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}{2} - \frac{\gamma\cdot\gamma}{2}\right) - \gamma\;\nabla_S\cdot\boldsymbol{u}_1 + \gamma\cdot\nabla_S\boldsymbol{u}_1 - \gamma\cdot\nabla_S\boldsymbol{u}_1\cdot\hat{\boldsymbol{n}}\,\hat{\boldsymbol{n}}. \tag{A 24}$$

Then substituting (A 16) and (A 24) into (A 15) we get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} + \gamma\;\nabla_S\cdot\boldsymbol{u}_1 = \gamma\cdot\nabla_S\boldsymbol{u}_1 - \hat{\boldsymbol{n}}\times\nabla\Phi + \nu\nabla\omega_1\cdot\hat{\boldsymbol{n}} - \nu\hat{\boldsymbol{n}}\cdot\nabla\omega_1 - \{\gamma\cdot\nabla\boldsymbol{u}_1\cdot\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}\cdot\nabla\boldsymbol{u}_1\cdot\gamma\}\,\hat{\boldsymbol{n}} \tag{A 25}$$

where

$$\Phi = -\left(\frac{\partial\phi_2}{\partial t} + \frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}{2} - \frac{\gamma\cdot\gamma}{2} + \frac{p_1}{\rho} + gy\right) \tag{A 26}$$

is essentially the same quantity as in (50). The bracketed term at the end of (A 25) is zero because shear stress is zero at the free surface (see (6)) and $\mathrm{d}S\ \nabla_S \boldsymbol{\cdot} \boldsymbol{u}_1 = \mathrm{d}(\mathrm{d}S)/\mathrm{d}t$, where $\mathrm{d}S$ is a material surface element on side 1 of the free surface, so one may write

$$\frac{1}{\mathrm{d}S}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma \mathrm{d}S) = \boldsymbol{\gamma} \boldsymbol{\cdot} \nabla_S \boldsymbol{u}_1 - \hat{\boldsymbol{n}} \times \nabla \Phi + \nu \nabla \boldsymbol{\omega}_1 \boldsymbol{\cdot} \hat{\boldsymbol{n}} - \nu \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla \boldsymbol{\omega}_1. \tag{A 27}$$

Now we use

$$\boldsymbol{\gamma} \boldsymbol{\cdot} \nabla_S \boldsymbol{u}_1 = \nabla_S \boldsymbol{\cdot} (\boldsymbol{\gamma}\ \boldsymbol{u}_1) - (\nabla_S \boldsymbol{\cdot} \boldsymbol{\gamma})\ \boldsymbol{u}_1 \tag{A 28}$$

$$= \nabla_S \boldsymbol{\cdot} (\boldsymbol{\gamma}\ \boldsymbol{u}_1) - \boldsymbol{\omega}_1 \boldsymbol{\cdot} \hat{\boldsymbol{n}}\ \boldsymbol{u}_1 \tag{A 29}$$

and write, finally,

$$\frac{1}{\mathrm{d}S}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma \mathrm{d}S) = \nabla_S \boldsymbol{\cdot} (\boldsymbol{\gamma}\ \boldsymbol{u}_1) - \hat{\boldsymbol{n}} \times \nabla \Phi + \nu \nabla \boldsymbol{\omega}_1 \boldsymbol{\cdot} \hat{\boldsymbol{n}} - \{\hat{\boldsymbol{n}} \boldsymbol{\cdot} \boldsymbol{\omega}_1\ \boldsymbol{u}_1 + \nu \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla \boldsymbol{\omega}_1\}. \tag{A 30}$$

(This equation may be derived from Wu (1995, equation 61*a*) upon interpreting his 'interface velocity' as $\boldsymbol{u}_1$ and calculating the acceleration on side 2 in terms of the material derivative on side 1 as we have done here.) The first three terms on the right can be reduced to line integrals when integrated over a surface patch, and therefore are surface 'divergence' terms, and the bracketed term is the flux of vorticity into this patch through the surface (see (A 34) below for this interpretation of vorticity flux); after integrating over a patch we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_1} \boldsymbol{\gamma} \mathrm{d}S = \int_{C_1} \hat{\boldsymbol{m}} \boldsymbol{\cdot} \boldsymbol{\gamma}\ \boldsymbol{u}_1 \mathrm{d}s - \int_{C_1} \Phi \hat{\boldsymbol{t}} \mathrm{d}s + \int_{C_1} \nu \nabla \boldsymbol{u}_1 \boldsymbol{\cdot} \hat{\boldsymbol{t}} \mathrm{d}s - \int_{S_1} (\hat{\boldsymbol{n}} \boldsymbol{\cdot} \boldsymbol{\omega}_1\ \boldsymbol{u}_1 + \nu \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla \boldsymbol{\omega}_1) \mathrm{d}S, \tag{A 31}$$

where S_1 is a material surface on the free surface, C_1 its boundary curve, $\hat{\boldsymbol{t}}$ is a tangent vector on C_1 in the direction of integration, and $\hat{\boldsymbol{m}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$ is the normal to C_1 in the surface.

In order to interpret the vorticity flux in (A 30) we note that Helmholtz's vorticity equation in the interior of the fluid,

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boldsymbol{\omega} \boldsymbol{\cdot} \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}, \tag{A 32}$$

has a vortex stretching term on the right-hand side which may be written

$$\boldsymbol{\omega} \boldsymbol{\cdot} \nabla \boldsymbol{u} = \nabla \boldsymbol{\cdot} (\boldsymbol{\omega}\ \boldsymbol{u}) \tag{A 33}$$

since the divergence of the vorticity is zero. Therefore, using the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{\omega} \mathrm{d}V = \int_S (\hat{\boldsymbol{n}} \boldsymbol{\cdot} \boldsymbol{\omega}\ \boldsymbol{u} + \nu \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla \boldsymbol{\omega}) \mathrm{d}S. \tag{A 34}$$

Thus we identify $\nu \nabla \boldsymbol{\omega}$ as a diffusive flux tensor and $\boldsymbol{\omega u}$ as a stretching flux tensor. (These are fluxes relative to the motion of the fluid. In general there is also a convective flux tensor, $-\boldsymbol{u\omega}$.) The surface vorticity flux is interpreted from (A 31) in a similar way. The flux of surface vorticity in the direction $\hat{\boldsymbol{m}}$ on the surface is $\hat{\boldsymbol{m}} \boldsymbol{\cdot} \boldsymbol{\gamma} \boldsymbol{u}_1 - \Phi \hat{\boldsymbol{t}} + \nu \nabla \boldsymbol{u}_1 \boldsymbol{\cdot} \hat{\boldsymbol{t}}$ with $\hat{\boldsymbol{t}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{m}}$. This can also be written as $\hat{\boldsymbol{m}}$ dotted with a flux tensor.

If the material volume V intersects the free surface in the surface S_1 of (A 31) then the flux of vorticity into the volume V across S_1 is equal to the flux of vorticity out of the vortex sheet across this part of the surface. If we extend the volume to infinity, so that it covers the entire fluid, the line integrals go to zero and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{fluid}} \boldsymbol{\omega}\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{intfc}} \boldsymbol{\gamma}\,\mathrm{d}S = 0. \tag{A 35}$$

Therefore total vorticity is conserved, as was to be shown.

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