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Improved Boundary Integral Method for Inviscid Boundary Condition Applications

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I. Introduction

THIS Note deals with the computation of the potential part of an unsteady, incompressible, viscous flow around an arbitrary configuration and the enforcement of the no-through flow boundary condition. To solve the full problem, it should be complemented by a solver to account for the convective and viscous part of the flow.

To generate the velocity potential, one often uses boundary integrals. The boundary may be considered as a surface of discontinuity[1] (vortex sheet) and is usually discretized with panels. The inviscid boundary condition (no through flow) is then applied in a Neumann or Dirichlet form on the streamfunction to determine the unknown vorticity distribution. However, the solution to this problem is not unique, and the additional constraint, of conservation of total circulation needs to be imposed. The resulting sets of equations are not numerically well-conditioned, and the accuracy of the solution deteriorates as the thickness of the body is decreased and/or the number of the panels increases. Here a rigorous approach is presented (similar to the one used in Ref. 2) involving the application of the internal Neumann boundary condition, in which the resulting system of equations is well-conditioned, thus permitting an efficient and accurate solution with direct or iterative matrix inversion techniques. The method does not increase the computational cost, and it is found to improve the conditioning of the system by several orders of magnitude, the improvement being more pronounced as the number of panels increases.

II. Governing Equations and Boundary Conditions

Consider a two-dimensional body, (s, n), in an incompressible viscous flowfield induced by a uniform flow and vorticity in the wake. The flow is governed by the Navier-Stokes equations, which may be expressed in terms of the vorticity ($\omega \hat{e}_z = \nabla \times u$) as:

$$\frac{\partial \omega}{\partial t} + u \cdot \frac{\partial \omega}{\partial x} = \nu \nabla^2 \omega \tag{1}$$

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along with the continuity equation ($\nabla \cdot u = 0$) and the no-slip boundary condition on the body. Using now the continuity equation along with the definition of the vorticity, the velocity u may be determined by the solution of

$$\nabla^2 u = -\nabla \times \omega \hat{e}_z$$

or in terms of the stream function

$$\nabla^2 \Psi = \omega \tag{2}$$

with $u = \nabla \times \Psi \hat{e}_z$. The stream function may be decomposed as

$$\Psi = \Psi_0 + \Psi_{\text{ext}}$$

where Ψ_{ext} is a particular solution of Eq. (2) and may be computed via the Biot-Savart law and Ψ_0 is the solution of the respective homogeneous equation and is chosen so that the velocity satisfies the kinematic boundary condition of no-through flow on the body (or $\Psi = \text{const}$). Once the velocity field has been calculated, then Eq. (1) needs to be solved to obtain the full solution of the problem. A number of numerical methods (finite differences, vortex methods, spectral methods) may be employed, but this analysis is beyond the scope of this Note.

Here we present an efficient scheme to solve for the potential part of the flow, namely the solution of

$$\nabla^2 \Psi_0 = 0$$

with $\Psi_0 + \Psi_{\text{ext}} = \text{const}$ on the body.

We consider Ψ_0 to arise because of a vortex sheet along the boundary of the body so that the flowfield at any point x in the domain is described by the stream function

$$\Psi(x) = -\frac{1}{2\pi} \oint \log|x - x(s')| \frac{d\Gamma}{ds'} ds' + \Psi_{\text{ext}}(x) \tag{3}$$

where the stream function $\Psi_{\text{ext}}(x)$ includes the contribution from the freestream and the vorticity in the wake and $d\Gamma/ds$ denotes the circulation distribution of the vortex sheet. Applying the inviscid boundary condition and the additional constraint that the total circulation ejected from the surface of the nonrotating body is zero, respectively, results in Eqs. (4) and (5) for the Dirichlet and internal Neumann boundary conditions:

$$c = -\frac{1}{2\pi} \oint \log|x(s) - x(s')| \frac{d\Gamma}{ds'} ds' + \Psi_{\text{ext}}[x(s)] \tag{4a}$$

$$\oint \frac{d\Gamma}{ds'} ds' + \Gamma_{\text{wake}} = 0 \tag{4b}$$

and

$$\frac{d\Gamma}{ds} - \frac{1}{\pi} \oint \frac{\partial}{\partial n} [\log|x(s) - x(s')|] \frac{d\Gamma}{ds'} ds' = -2 \frac{\partial \Psi_{\text{ext}}}{\partial n} [x(s)] \tag{5a}$$

$$\oint \frac{d\Gamma}{ds'} ds' + \Gamma_{\text{wake}} = 0 \tag{5b}$$

These sets of equations may be solved to any accuracy by a panel method.[3] Discretizing the body with M vortex panels results in a system of equations for the M unknown strengths. The two linear sets of equations may be expressed in matrix form:

$$Kf = g \quad \text{(Dirichlet boundary condition)}$$

$$Gf = h \quad \text{(Neumann boundary condition)}$$

System $Kf=g$ has $M+1$ equations with $M+1$ unknowns (the $M+1$th unknown being the constant c). However, for thin bodies or bodies with cusped trailing edges, two panels can be very close to each other while corresponding to opposite normals. This would result[4] in the matrix K having two nearly identical rows and therefore being nearly singular. System $Gf=h$ has $M+1$ equations but with only M unknowns, the strengths of the panels. To solve this latter system of equations, several approaches are plausible[3,5]: source method, least squares solution, introduction of a new unknown, elimination of one equation, etc.

However, these approaches rely mainly on empirical criteria, which become more important than any details of the numerical implementation. Moreover, in the case of thin bodies, two panels can be separated by a distance much smaller than either length (e.g., in high curvature trailing edges), resulting in the existence of two nearly equal but opposite signed rows in the matrix G and producing an ill-conditioned system of equations.[4] As was suggested by a reviewer, in practice this anomaly is more pronounced for source methods because the solution source strength increases without bounds as the body thickness goes to zero. Our method consists of an application of the Fredholm alternative to the solution of Eqs. (5). This results in a system of equations that is well behaved, independent of the thickness of the body and the size of the panels and therefore improving the accuracy of the calculations, especially when large numbers of panels are necessary for the description of the body.

III. Theoretical Formulation

By setting

$$G(s,\ s') = \frac{1}{\pi}\frac{\partial}{\partial n}\,[\log|x(s)-x(s')|]$$

$$f(s) = \frac{\mathrm{d}\Gamma}{\mathrm{d}s}\,\mathrm{d}s \qquad h(s) = -2\,\frac{\partial \Psi_{\mathrm{ext}}}{\partial n}\,[x(s)]$$

Eq. (5a) may be expressed equivalently as

$$f(s) - \oint G(s,\ s')\,f(s')\ \mathrm{d}s' = h(s) \tag{6}$$

Equation (6) is an integral equation of the second kind with a nontrivial homogeneous solution. According to the Fredholm alternative,[6] a solution to Eq. (6) exists if

$$\oint h(s')\psi(s')\ \mathrm{d}s' = 0$$

where $\psi(s)$ is the solution of the eigenvalue problem for the transposed equation with $\lambda=1$; i.e.,

$$\oint G(s',\ s)\psi(s')\ \mathrm{d}s' = \lambda\psi(s) \tag{7}$$

For the kernel $G(s,\ s')$ considered, $\psi(s)=\mathrm{const}$ is the solution for $\lambda=1$, so the necessary condition is

$$\oint h(s')\ \mathrm{d}s' = 0$$

This is indeed the case, as the preceding condition is equivalent to

$$\oint u_{\mathrm{ext}}\cdot \mathrm{d}l = 0$$

where $u_{\mathrm{ext}} = \nabla\times(\Psi_{\mathrm{ext}}\hat{e}_z)$. The solution of Eq. (6) is not unique, since if $f_0(s)$ is a solution then an arbitrary number of solutions $f(s)$ of the form

$$f(s) = f_0(s) + \alpha\phi(s)$$

may be obtained, where α is an arbitrary constant and $\phi(s)$ the solution of the following eigenvalue problem for $\lambda=1$:

$$\oint G(s,\ s')\phi(s')\ \mathrm{d}s' = \lambda\phi(s) \tag{8}$$

However uniqueness of the solution for our problem is guaranteed by the the application of Kelvin's theorem (5b):

$$\oint f(s)\ \mathrm{d}s = -\Gamma_{\mathrm{wake}} \tag{9}$$

or equivalently, α is uniquely determined by:

$$\alpha = -\Gamma_{\mathrm{wake}} + \frac{\oint f_0(s)\ \mathrm{d}s}{\oint \phi(s)\ \mathrm{d}s} \tag{10}$$

IV. Proposed Scheme

An alternative way for the application of the inviscid boundary condition is proposed here based on the spectral decomposition of the kernel $G(s,\ s')$. This kernel may be decomposed as

$$G(s,\ s') = \sum_{i=1}^{\infty}\lambda_i\phi_i(s)\psi_i(s') \tag{11}$$

with λ_i, $\phi_i(s)$, eigenvalues and eigenfunctions given by the solution of Eq. (8), and $\psi_i(s)$ eigenfunctions of the transposed kernel given by the solution of Eq. (7). These eigenfunctions are normalized so that Eqs. (7) and (8) are valid for the decomposed kernel defined in Eq. (11). Hence

$$\oint \phi_i(s)\psi_j(s)\ \mathrm{d}s = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{12a}$$

and

$$\psi_1(s) = \mathrm{const} = 1/L \tag{12b}$$

where L is the perimeter of the shape considered. Note that the eigenfunction $\phi_1(s)$ corresponding to the eigenvalue $\lambda_1=1$ is a solution of Laplace's equation satisfying the inviscid boundary condition in the absence of any external flowfield.

Now the condition imposed by Kelvin's theorem may be substituted by an equivalent one, by multiplying both sides of Eq. (9) with $\lambda_1\phi_1(s)\psi_1(s')$, so that the new set of equations is

$$f(s) - \oint G(s,\ s')f(s')\ \mathrm{d}s' = h(s) \tag{13a}$$

$$\frac{\phi_1(s)}{L}\oint f(s')\ \ \mathrm{d}s' = -\frac{\phi_1(s)}{L}\,\Gamma_{\mathrm{wake}} \tag{13b}$$

Adding Eq. (13b) to Eq. (13a) results in the final form of the integral equation that needs to be solved for the unknown function $f(s) = \mathrm{d}\Gamma/\mathrm{d}s$:

$$f(s) - \oint\left[G(s,\ s') - \frac{\phi_1(s)}{L}\right]f(s')\ \mathrm{d}s' = h(s) - \frac{\phi_1(s)}{L}\,\Gamma_{\mathrm{wake}} \tag{14}$$

In this form the new kernel is just the kernel given by Eq. (11) but with the first term in the spectral decomposition eliminated. Hence, the singularity associated with the kernel $G(s, s')$ is annihilated and Eq. (14) is well posed. This equation is solved using panel methods, and the resulting system of equations is a well-conditioned one. Note that in the preceding formulation of the problem it is necessary to know a priori the form of the eigenfunction $\phi_1(s)$, which is not available in general for arbitrary shapes. For elliptic bodies (including the cylinder and the flat plate), this eigenfunction was determined analytically and is given as a function of the eccentricity ϵ of the ellipse and the polar angle θ:

$$\phi_1(\theta) = \left[1-\epsilon\cos^2(\theta)\right]^{-1/2}$$

For arbitrary configurations (e.g., a NACA0012 airfoil), $\phi_1(s)$ may be obtained by solving the eigenproblem (8) using panel methods. The eigenfunction is then obtained using as collocation points the eigenvector of the resulting matrix. This adds to the computational cost of the method, but this eigenfunction needs to be computed only once for the considered shape. For the case of a cylinder, $\epsilon = 0$, and so the unknown function $f(s)$ may be obtained directly as Eq. (14) reduces to

$$f(s) = h(s) - (1/L)\Gamma_{\text{wake}} \tag{15}$$

In Fig. 1 the eigenvalues for two ellipses are shown. Calculations for various ellipses exhibit an interesting oscillatory behavior of the eigenvalues and a broadening of the spectrum as the thickness of the body decreases.

V. Numerical Application—Results

To compare the various methods for the application of the boundary conditions, the possibility of ill-conditioning in the numerical solution to the resulting linear systems of equations was examined. This ill-conditioning is in the sense that the results of the computation depend continuously on the data with a proportionality constant that is not very large.[7] That is, considering the system $Ax = b$, small changes in b result in large changes in x. Conversely, the underlying requirement for a well-conditioned system is that a small perturbation of the position of the vortices in the wake of the body resulting in a small change δb on the right-hand side, should not result in a significant change δx on the results of the computation. As a measure of this change, we consider the condition index κ of the matrix A defined as

$$\kappa = \kappa(A) = \|A^{-1}\| \, \|A\|$$

so that

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa \frac{\|\delta b\|}{\|b\|}$$

In the present calculations of κ, the 1-norm[7] has been implemented.

In Fig. 2 and Fig. 3, the condition index for the system resulting from the application of the Dirichlet boundary condition and the condition index of the system resulting from the proposed scheme are shown for the potential flow around ellipses of different thickness and a NACA0012 airfoil, respectively.

As can be seen from these figures, as the number of panels is increased and the thickness of the body is decreased the Dirichlet boundary condition results in a system that is ill-conditioned. For small numbers of panels, using computers with 8 or 10 significant figures, this anomaly does not seriously affect the results. When large numbers (in the order of thousands) of panels are necessary for the description of the boundary, this ill-conditioning becomes important. For example, in the context of using vortex methods for the description of the flow, in order to obtain accurate results one must always keep the ratio of the distance of the vortex from the panel to the length of the panel less than unity. So when boundary layers are resolved with vortices, their small distance from the body dictates the use of very large numbers of panels. However, this would result in higher computer cost and would adversely affect the accuracy of the solution. On the other hand, the application of the present scheme results in a very well-conditioned system of equations with a condition index that tends asymptotically to a small value as the number of panels is increased.

The advantage of the present formulation is more pronounced as the number of the panels is increased and/or the thickness of the body decreases. When we solve the system of equations by direct elimination, the conditioning (and the size) of the system is such that it does not pose any numerical difficulty. However, as the number of unknowns increases, direct methods become prohibitive, and we have to rely on iterative techniques such as conjugate gradient. The conditioning of the coefficient matrix is a key factor to the convergence of these techniques, and, as may be observed from the preceding figures, the present formulation is advantageous for such computations.

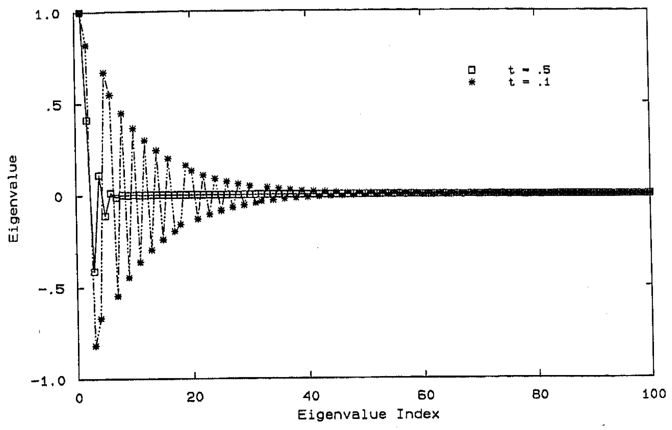

Fig. 1 Eigenvalues of ellipses having thickness t: 50%, 10%.

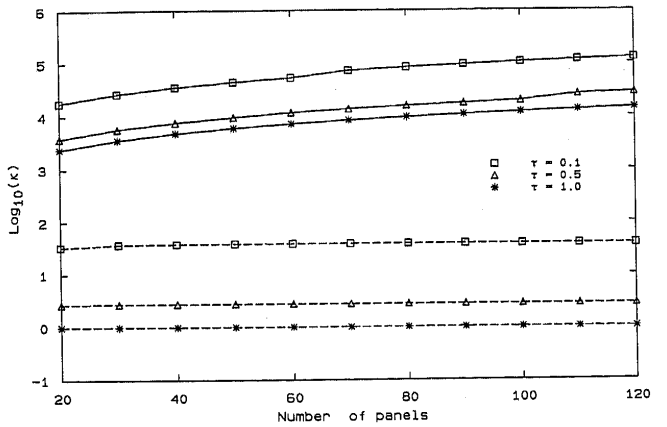

Fig. 2 Condition index for ellipses of different thickness τ resulting from the classical approach (solid lines) and the proposed method (dashed lines).

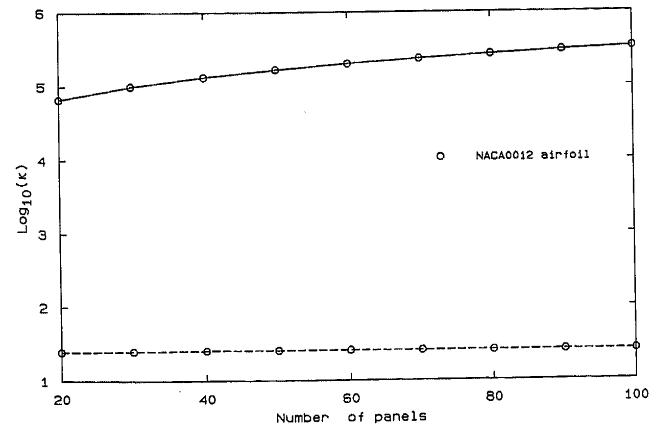

Fig. 3 Condition index for a NACA 0012 airfoil resulting from the classical approach (solid line) and the proposed method (dashed line).

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Computation of Unsteady Supersonic Quasi-One-Dimensional Viscous-Inviscid Interacting Internal Flowfields

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Introduction

THE study and analysis of internal flows have received significant attention over the past several decades because the operation of many physical devices, particularly regarding aerospace-related hardware, depends on proper designs to achieve near-optimum operating characteristics. Examples of such devices include any configuration where the flow is confined and an exchange between pressure and kinetic energy is desired (e.g., engine inlets, wind tunnel diffusers, rocket nozzles, etc.). These devices can be geometrically complex as well as viscous-flow dominated. Moreover, certain configurations and conditions can result in unsteady flow.

In the past, the design of these devices has, for the large part, depended on empirically based methodologies. More recently, computational techniques have played an increasingly important role in the design process as hardware becomes less conservative and is required to operate near the edge of the design envelope. A thorough computational investigation of flowfields of this type requires solution of the full Reynolds-averaged, multidimensional, time-dependent Navier-Stokes equations. Because solutions to these equations provide essentially all pertinent flowfield parameters (assuming these solutions are of acceptable accuracy), it is possible to perform parametric studies of a proposed geometry/flowfield combination, which could, in turn, be used to reduce significantly the risk associated with new hardware design. Unfortunately, obtaining numerical solutions to these equations for complex geometries and unsteady flowfields is expensive and time consuming, even using today's largest and fastest supercomputers. Therefore, it is important to seek alternative means of performing compute-based studies of proposed new hardware designs.

The development of an engineering tool through which preliminary estimates of unsteady supersonic internal flow processes can be generated using available workstation-based hardware is the underlying mission of the present effort. The approach encompasses a viscous-inviscid interaction technique where the flowfield is separated into inviscid and viscous parts and the appropriate equation sets (corresponding to each region) are solved using a new interaction technique. The assumption is made that inviscid flow phenomena can be adequately represented using the unsteady, quasi-one-dimensional (Q1D) Euler equations, whereas viscous effects are accounted for using integral boundary-layer equations for unsteady, two-dimensional, turbulent flow. Under these assumptions, the objective of this Note is to demonstrate a new method for computing unsteady interactions whereby the inviscid and viscous equations are cast as a (single) coupled system of partial differential equations and solved simultaneously. This approach to computing unsteady viscous-inviscid interactions for internal flows differs from techniques previously reported, where each equation set is solved independently of the other; i.e., the coupling procedure was performed between the *solutions* of the equations *and not between the equations themselves*. Viscous-inviscid coupling at the equation level has been demonstrated by Drela and Giles[1] for two-dimensional laminar/turbulent airfoil flows, and excellent results were obtained using minimal computational resources.[1] However, the technique reported in Ref. 1 involved steady external flows, and the present effort attempts to apply the approach to unsteady internal configurations. The motivation behind the approach presented herein is the observation that coupling the solutions results in schemes that can have convergence difficulties.[2] Moreover, marching the equations simultaneously yields a time-accurate coupling for unsteady flows which is more straightforward than marching each equation set separately.[3] However, it must be emphasized that the validity of using the simplified equations is very problem dependent and, similar to other analytical/computational techniques, requires experience and engineering judgment with regard to whether the approach and/or computed solutions represent reality. No attempt is made at quantifying specific classes of problems for which the approach presented herein can be used.

Use of the Q1D assumption for internal flow analysis is certainly not new, and both viscous and inviscid techniques have been reported. For example, the method of Mays[4] was perhaps the first to investigate time-dependent, variable-area duct flows by numerically solving the unsteady, Q1D inviscid flow equations. For purely steady subsonic diffuser flows, Harsha and Glassman[5] developed a technique whereby the boundary-layer equations for turbulent flow were solved in a marching fashion by iterating on the pressure gradient required to satisfy mass conservation. White and Anderson[6] investigated steady, inviscid, Q1D nozzle flows by numerically solving the Euler equations. Adams et al.[7] and Varner et al.[8] solved the Q1D inviscid equations for supersonic mixed compression inlets using a split-characteristics approach designed to capture accurately moving shocks for investigating inlet unstart/restart. A technique similar to that presented herein was reported by Bussing and Murman,[9] where the unsteady Q1D Euler equations were solved by dividing the flow into viscous and inviscid regions and modifying the inviscid equations to account for the mass defect brought about by the presence of the boundary layer. However, boundary-layer parameters were computed in a separate program and consequently were not solved alongside the inviscid equations.

A generic geometric configuration for which the present analysis is applicable is shown in Fig. 1. Fundamental considerations to the approach taken herein are the assumptions that the flowfield within the confines of the channel is not fully developed and that an inviscid core of fluid exists and is allowed to interact with the viscous region near the wall. The displacement of mass brought about by the presence of this viscous region has a thickness of δ^*, defined by

$$\rho_e u_e \delta^* = \int_0^\infty (\rho_e u_e - \rho u)\, \mathrm{d}y$$

This displacement of mass affects the available flow area and consequently is a primary mechanism through which the viscous-inviscid interaction proceeds.

It should be pointed out that the preceding expression for δ^* is exact for planar flow but is in error for axisymmetric flow. However, the preceding expression approaches the true mass-flow defect length for the axisymmetric case when the local body radius is large compared to the local boundary-layer thickness.[10] This is an important distinction[11] because for cases

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