Fluid-Structure Interaction with Vortex methods and the Finite Element Method

Siddhartha Verma
Gengyun Li, Petros Koumoutsakos

October 5, 2016
Fluid-Structure Interaction

- **Engineering applications**
  - Wind turbines, aircraft frames, buildings, bridges, bio-inspired devices…

- **Biomedical applications**
  - Artificial heart valves, stents, aneurysms…

- **Simulations difficult with body fitted grids**
  - Complex shapes, deforming geometries, moving meshes - *large overhead*

- **Vortex methods + Brinkman penalisation**
  - Fast & accurate, parallelizes easily
  - Multiple deforming geometries with relative ease
Objective: Combine vortex methods and FEM

- Vortex methods + Brinkman penalization
  - Fluid velocity penalized to match $u_s$ at appropriate grid nodes
  - Solid influences fluid

- Finite Element Method
  - Compliant solids deform due to flow-induced forces on the solid boundary
  - Fluid influences solid via surface-forces $f_{ext}$

- Weak coupling (partitioned approach)
  - Staggered stepping of fluid- and solid-solvers

Vortex Methods

$\frac{D\omega}{Dt} = \nu \nabla^2 \omega + \lambda \nabla \times (\chi (u_s - u))$

Finite Element Method

$M \ddot{v} + C \dot{v} + K v = f_{ext}$
Fluid Mechanics
(Vortex Methods)
Background

- Enforcing Boundary Condition (BC) on complex, deforming geometries
- Arbitrary Lagrangian Eulerian (ALE)
  - Body Fitted grid: Difficulty with Multiple bodies & with Deforming geometries
  - Expensive: frequent re-meshing
- Immersed Boundary (IB) method
  - Simple Cartesian grids
  - BCs enforced via Dirac-delta forcing

Vortex methods & Brinkman penalization

- Remeshed vortex methods
- Solve vorticity form of incompressible Navier-Stokes
  
  \[
  \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} + \lambda \chi (\mathbf{u}_s - \mathbf{u})
  \]


- Brinkman penalization
  - Relatively easy to implement, computationally cheap
  - Accounts for fluid-solid interaction
  - Penalty term enforces no-slip BC

  \[
  \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega + \lambda \nabla \times (\chi (\mathbf{u}_s - \mathbf{u}))
  \]

  Angot et al., Numerische Mathematik (1999)

- Single grid handles both Fluid and Solid
  - Fluid: \( \chi = 0 \)
  - Solid: \( \chi = 1 \)

Hejlesen et al., JCP (2015)
Vortex methods & Brinkman penalization

- Remeshed vortex methods
- Solve vorticity form of incompressible Navier-Stokes


\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} + \lambda \chi (\mathbf{u}_s - \mathbf{u}) \]

**ADVANTAGES:**

1) Do advection only where vorticity is non-zero (think uniform flow, where FV/FD have to do computations in non-important regions)
2) Time step not limited by CFL (i.e., stability issues, but rather limited by accuracy concerns)
3) SL schemes tend to exhibit lower numerical diffusion
Implementation

- Wavelet-based adaptive grid
- Cost-effective compared to uniform grids
- Each thread receives a small data block (up to 60% CPU utilisation)
- Open source code: available on github
  - [https://github.com/cselab/MRAG-I2D](https://github.com/cselab/MRAG-I2D)
  - Rossinelli et al., JCP (2015)

- Godunov splitting
  1. Penalization
  2. Diffusion
  3. Advection

\[
\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega + \lambda \nabla \times (\chi (\mathbf{u}_s - \mathbf{u}))
\]

\[
\begin{align*}
\omega^* &= \omega^n + \Delta t \left\{ \lambda \nabla \times (\chi (\mathbf{u}_s - \mathbf{u})) \right\} \\
\omega^{**} &= \omega^* + \Delta t \left\{ \nabla^2 \omega^* \right\}
\end{align*}
\]

\[
\frac{D \omega^{**}}{Dt} = 0 \quad \Rightarrow \quad \omega^{n+1} = \omega^{**}
\]
Surface forces

• Force computations require the stress tensor $\sigma$
  
  • Drag = net force opposing velocity
  
  • Compute surface traction, dot product with velocity vector
  
  • $+ve \Rightarrow$ Thrust; $-ve \Rightarrow$ Drag

• Stress tensor requires two things
  
  • Pressure
  
  • Strain-rate tensor

• Vorticity form of Navier Stokes
  
  • Advantage: no need to deal with pressure
  
  • Disadvantage: pressure term unavailable!

\[ F_{surf} = \int \sigma \cdot nds \]
\[ F_{D/T} = \frac{F_{surf} \cdot u}{|u|} \]
\[ \int -Pn \cdot dS \]
\[ D = \frac{1}{2} (\nabla u + \nabla u^T) \]
\[ \int 2\mu D \cdot ndS \]
\[ \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega + \lambda \nabla \times (\chi (\mathbf{u}_s - \mathbf{u})) \]
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} + \lambda \chi (\mathbf{u}_s - \mathbf{u}) \]
Computing pressure

• For Pressure: divg. of the N-S equans

\[ \nabla \cdot \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} + \lambda \chi (\mathbf{u}_s - \mathbf{u}) \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \mathbf{u}^T : \nabla \mathbf{u} = -\frac{\nabla^2 P}{\rho} + \frac{1}{\rho^2} \nabla \rho \cdot \nabla P + \nabla \cdot (\lambda \chi (\mathbf{u}_s - \mathbf{u})) \]

\[ \nabla^2 P = \rho (-\nabla \mathbf{u}^T : \nabla \mathbf{u} + \nabla \cdot (\lambda \chi (\mathbf{u}_s - \mathbf{u}))) \]

• Poisson’s equation: solved using free-space Green’s function

\[ \nabla^2 G = \delta(x - x_0) \]

\[ \nabla G = 0 \quad \text{on } dS \]

\[ \Rightarrow G = \frac{1}{2\pi} \ln|r| \]

\[ P = G \ast \text{RHS} \]

• Strain-rate tensor (for frictional forces)

\[ D = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]

STRESS that Pressure solve is not done on the entire domain, but a very small subset consisting of the solid surface points.
The Fast Multipole Method (FMM)

- Extremely fast solutions for the Poisson equation ('N-body' problem)
  - Speed-up: 10,000X and more

- Speed-up results from combination of
  - Tree-code: groups together sources into clusters
  - Truncated series expansions

\[
P(x) = \int G(x, x') \left( -\rho \left( \nabla u^T \cdot \nabla u \right) + \rho \lambda' \nabla \cdot \left( \chi (u_s - u) \right) \right) dx'
\]

\[
P(x) = \sum_{x'} G(x, x') f(x') h^2(x')
\]

\[
P(z) = \frac{S}{2\pi} \Re \{ \log(z) \} - \sum_{k=1}^{p} \Re \left\{ \frac{\alpha k}{z^k} \right\}
\]
Flow around a cylinder

- Impulsively started flow around a stationary cylinder
- Spatial distribution of surface-pressure & drag time-evolution agree with reference data

Surface vorticity distribution

$Re = 550$

$Re = 1000$
Self-propelled swimmer \((Re=4000)\)

- Body-undulations displace fluid
- Reaction forces propels swimmer forward

Surface-force vectors

\(T = 0.05\)
Validation for self-propelled swimmer

- Compare unsteady acceleration computed using surface-forces to penalisation accel.
  \[ a_{\text{penal}}^n = \frac{u_{CM}^{n+1} - u_{CM}^{n-1}}{2\Delta t} \quad a_{SF}^n = \frac{F_P^n + F_V^n}{M} \]

- Good agreement: Surface-force computation accurate for complex, temporally deforming bodies

- Low sensitivity to grid-resolution
Solid Mechanics

(now ‘$u$’ = displacement!)
FEM Steps : Pre-processing

1. Mesh the object (Delaunay triangulation)
   - Equi-angular triangles (few ‘skinny’ triangles)
   - No vertex lies inside the circumcircle of other triangles
   - Returns list of element connectivity

2. Assign degrees of freedom
   - Each node has 2DoFs: $[u, v]$ (displacements in x and y)

\[
M \ddot{u} + C \dot{u} + Ku = f_{ext}
\]

- $u$: Displacement
- $M$: Mass matrix - inertia
- $K$: Stiffness matrix - elasticity

Credit: http://www.cescg.org/
FEM Steps: Generate system matrices

3. Generate local ‘mass’ and ‘stiffness’ matrices for each CST element

\[ K_e = t_e A_e (B' DB) \]

\[ M_e = \int \int \int \rho N^T N \, dV \]

4. Assemble local element matrices into a global system matrix

- Each node may belong to multiple elements
FEM Steps: Constrain system & Solve

5. Apply constraints to fix object in space
   - Remove rows and columns corresponding to known displacements

\[
\begin{bmatrix}
    k_{11} & k_{12} & k_{13} & k_{14} & \cdots \\
    k_{21} & k_{22} & k_{23} & k_{24} & \cdots \\
    k_{31} & k_{32} & k_{33} & k_{34} & \cdots \\
    k_{41} & k_{42} & k_{43} & k_{44} & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
    u_2 \\
    u_3 \\
    u_4 \\
    \vdots
\end{bmatrix}
+ \begin{bmatrix}
    m_{22} & m_{23} & m_{24} & \cdots \\
    m_{32} & m_{33} & m_{34} & \cdots \\
    m_{42} & m_{43} & m_{44} & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
    \ddot{u}_2 \\
    \ddot{u}_3 \\
    \ddot{u}_4 \\
    \vdots
\end{bmatrix}
= \begin{bmatrix}
    f_2 \\
    f_3 \\
    f_4 \\
    \vdots
\end{bmatrix}
- \begin{bmatrix}
    k_{21} \\
    k_{31} \\
    k_{41} \\
    \vdots
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots
\end{bmatrix}
- \begin{bmatrix}
    m_{21} \\
    m_{31} \\
    m_{41} \\
    \vdots
\end{bmatrix}
\dddot{u}_1
\]

\[\begin{align*}
\ddot{u}^{n+1} &= 2\left(\frac{u^{n+1} - u^n}{\Delta t}\right) - \ddot{u}^n \\
\dddot{u}^{n+1} &= 4\left(\frac{u^{n+1} - u^n}{\Delta t^2}\right) - 4\frac{\ddot{u}^n}{\Delta t} - \dddot{u}^n
\end{align*}\]

6. Advance in time using implicit integration scheme (Newmark’s method)

\[\mathbf{M}\ddot{u} + \mathbf{C}\dot{u} + \mathbf{K}\mathbf{u} = \mathbf{f}_{ext}\]

- \(\mathbf{u}\): Displacement
- \(\mathbf{M}\): Mass matrix - inertia
- \(\mathbf{K}\): Stiffness matrix - elasticity

RHS known
Objective: Combine vortex methods and FEM

- Vortex methods + Brinkman penalization
  - Fluid velocity penalized to match $u_s$ at appropriate grid nodes
  - Solid influences fluid

- Finite Element Method
  - Compliant solids deform due to flow-induced forces on the solid boundary

- **OOFEM**: Open source FEM solver
  
  B. Patzák, OOFEM project home page
  
  http://www.oofem.org, 2000
Coupling the Fluids solver with FEM

Vortex methods + Brinkmann penalization

\[
\nabla^2 \psi^n = -\omega^n \\
u^n = \nabla \times \psi^n
\]

\[
\rho^n = \rho_f + \sum_{j=1}^{N} (\rho^n_j - \rho_f) \chi^n_j \]

\[
u^n_i = \frac{1}{m_i} \int_{\Omega} \rho^n \chi_i^n \mathbf{u} \, dx \]

\[
\hat{\theta}_i^n = J_{1^{-1}} \int_{\Omega} \rho^n \chi_i^n (\mathbf{x} - \mathbf{x}_i^n) \times \mathbf{u} \, dx \]

\[
u_i^{n+1} = \hat{\theta}_i^n \times (\mathbf{x} - \mathbf{x}_i^n) \]

\[
\mathbf{u}_\lambda = \frac{\mathbf{u} + \lambda \chi \delta t \mathbf{u}_S}{1 + \lambda \chi \delta t} \quad \omega_\lambda = \omega^n + \nabla \times (\mathbf{u}_\lambda - \mathbf{u}^n) \]

\[
\nabla^2 P = \rho (-\nabla \mathbf{u}^2 : \nabla \mathbf{u} + \nabla \cdot (\lambda \chi (\mathbf{u}_s - \mathbf{u}))) \]

\[
D = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]

\[
\frac{\partial \omega_\lambda^n}{\partial t} = \nu \nabla^2 \omega_\lambda^n \]

\[
\frac{\partial \omega_\lambda^n}{\partial t} + \mathbf{u}_\lambda^n \cdot \nabla \omega_\lambda^n = 0 \]

\[
\omega^{n+1} = \omega^n + \Delta t \nabla \omega_\lambda^n \]

\[
x_i^{n+1} = x_i^n + u_i^{n+1} \Delta t^n, \quad \theta_i^{n+1} = \theta_i^n + \dot{\theta}_i^n \Delta t^n \]

Surface forces

\[
r^2 P = \mathbf{\cdot} \mathbf{u} + \nabla \times (\lambda \chi (\mathbf{u}_s - \mathbf{u})) \]

\[
D = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]

\[
\frac{\partial \omega_\lambda^n}{\partial t} = \nu \nabla^2 \omega_\lambda^n \]

\[
\frac{\partial \omega_\lambda^n}{\partial t} + \mathbf{u}_\lambda^n \cdot \nabla \omega_\lambda^n = 0 \]

\[
\omega^{n+1} = \omega^n + \Delta t \nabla \omega_\lambda^n \]

\[
x_i^{n+1} = x_i^n + u_i^{n+1} \Delta t^n, \quad \theta_i^{n+1} = \theta_i^n + \dot{\theta}_i^n \Delta t^n \]

RK2+LTS

RK2+LTS / particles

Explicit Euler
Coupling the Fluids solver with FEM

Velocity

\[ \nabla^2 \psi^n = -\omega^n \]
\[ \mathbf{u}^n = \nabla \times \psi^n \]

Projection

\[ \rho^n = \rho_f + \sum_{i=1}^{N} (\rho_i^n - \rho_f) \chi_i^n \]
\[ u_i^{n} = \frac{1}{m_i} \int_{\Omega} \rho_i^n \chi_i^n \mathbf{u}^n \, dx \]
\[ \dot{\mathbf{u}}_i^n = J^{-1} \int_{\Omega} \rho_i^n \chi_i^n (\mathbf{x} - \bar{x}_i^n) \times \mathbf{u}^n \, dx \]
\[ u_i^{n+1} = \dot{u}_i^n \times (\mathbf{x} - \bar{x}_i^n) \]

Penalization

\[ u_{\lambda} = \frac{\mathbf{u}^n + \lambda \chi \mathbf{u}_S}{1 + \lambda \chi \delta t} \]
\[ \omega_{\lambda} = \omega^n + \nabla \times (u_{\lambda} - \mathbf{u}^n) \]

Surface forces

\[ \nabla P = \rho (-\nabla \mathbf{u}^T : \nabla \mathbf{u} + \nabla \cdot (\lambda \chi (\mathbf{u}_S - \mathbf{u}))) \]
\[ D = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]

Diffusion

\[ \frac{\partial \omega_{\lambda}^n}{\partial t} = \nu \nabla^2 \omega_{\lambda}^n \]

Advection

\[ \frac{\partial \omega_{\lambda}^n}{\partial t} + u_{\lambda}^n \cdot \nabla \omega_{\lambda}^n = 0 \]
\[ \omega_{\lambda}^{n+1} = \omega_{\lambda}^n \]
\[ \mathbf{x}_i^{n+1} = \mathbf{x}_i^n + u_i^n \Delta t^n, \quad \theta_i^{n+1} = \theta_i^n + \hat{\theta}_i^n \Delta t^n \]

Force distribution

Deformable mesh

Deformed configuration

RK2+LTS

Explicit Euler

RK2+LTS / particles

OOOFEM

Solve
FSI: Dynamic tests

- Cantilever beam (fixed at lower end)
  - Uniform inflow
    \[ U_\infty = 0.1 \quad Re = 2000 \]
  - Flow induced force largest in the centre (high pressure due to flow stagnation)
- Beam rigid up until 1s, then allowed to deform
- Beam deformation
  - abrupt initially: impulsive force
  - then exhibits ‘wobbling’

Material: Biological Tissue equivalent
## Stiffer beams

<table>
<thead>
<tr>
<th>Material</th>
<th>Initial State</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiff rubber</td>
<td>Rigid</td>
<td>0.00s</td>
</tr>
<tr>
<td>Lead</td>
<td>Rigid</td>
<td>0.00s</td>
</tr>
</tbody>
</table>

[Diagram showing the comparison of stiffer beams]
Vortex induced vibrations

- Solid cylinder with attached tail
  - Tail rigid until steady state attained
    \[ U_\infty = 2.0 \quad Re = 200 \]
  - Exhibits large-amplitude oscillations

- Still to do
  - Quantitative comparisons with FSI benchmarks

Turek et al. (2010), Fluid Structure Interaction II, LNCS

Lee & You (Phys. Fluids, 2013)
Summary

- Method for determining surface pressure and shear forces
  - Using Vortex methods and Brinkman penalisation
- Validated using simulations of rigid and deforming objects
- Coupled with the Finite Element Method
  - Flow-induced deformation: cantilever beams
  - Vortex-induced oscillation of flexible tail

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \nabla^2 \omega + \lambda \nabla \times (\chi (u_s - u))
\]

\[
\nabla^2 P = \rho \left( -\nabla^T : \nabla u + \nabla \cdot (\lambda \chi (u_s - u)) \right)
\]

\[
D = \frac{1}{2} \left( \nabla u + \nabla^T u \right)
\]
Backup
Application: ‘Smart’ follower learning to maximise efficiency

- ‘Siphoning’ energy from the leader’s wake
- Follower autonomously learns optimal manoeuvres for maximising efficiency
  - Reinforcement Learning
  - Relatively long occurrences of $\eta=1$
  - Follower effectively hitching a ‘free ride’
- Overall, 28% gain in average efficiency
  - Energetic savings of 64%

$$R_\eta = \frac{T_u}{T_u + \max(P_{def}, 0)}$$
Variation of all three parameters

\[ \eta = \frac{P_{\text{Thrust}}}{P_{\text{Thrust}} + P_{\text{Def}}} \]
Spatial & temporal evolution

$T = 10.00$

$C_p$(Thrust)

$C_p$(Def)
Spatial & temporal evolution

- Max $\eta$ - neck and tail no drag, tail tip barely moving, fore section low curvature
- Min $\eta$ - tail tip max velocity, neck substantial drag
- Max Power - majority of the body high vel

Max $\eta$  \hspace{1cm} T = 8.21

Min $\eta$  \hspace{1cm} T = 8.57
Def. power along the body

- Majority of effort at tail
- Head: suction peak results in large thrust, with almost no effort
- Rest of body: thrust closely follows deformation power
Thrust generation along the body

- Majority of thrust from mid-body & tail
  - Head: drag caused by stagnation pressure. Thrust at bulge (suction peak)
  - ‘Neck’ region: no thrust, negligible drag
  - Tail end: Mostly drag!
Pressure FMM

- Poisson equation: solved using free-space Green’s function

\[ \nabla^2 G = \delta(x - x_0) \quad \Rightarrow \quad G = \frac{1}{2\pi} \ln|r| \]

\[ \nabla G = 0 \quad \text{on } dS \quad \Rightarrow \quad P = G \ast RHS \]

\[ \phi(x, y) = \sum_{j=1}^{m} m_j \log r \]

\[ \log(z - z_j) = \log r + \log(e^{i\theta}) = \log r + i\theta = \log|z - z_j| + i\theta \]

\[ \text{Re}(\log(z - z_j)) = \log|z - z_j| \]

So solution is the real part of \( \phi(z) = \sum_{j=1}^{m} m_j \log(z - z_j) \)
Pressure FMM

\[
\phi(z) = \sum_{j=1}^{m} m_j \log(z - z_j) = \sum_{j=1}^{m} m_j \left[ \log z + \log \left( 1 - \frac{z_j}{z} \right) \right] = M \log z + \sum_{j=1}^{m} m_j \sum_{l=1}^{\infty} \frac{(z_j)^l}{l} =
\]

\[
= M \log z + \sum_{l=1}^{\infty} \sum_{j=1}^{m} m_j \frac{(z_j)^l}{l} = M \log z + \sum_{l=1}^{\infty} \left[ z^{-l} \sum_{j=1}^{m} m_j \frac{z_j^l}{l} \right] = M \log z + \sum_{l=1}^{\infty} \frac{\alpha_l}{z^l}
\]

\[
\phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k}
\]

\[
\phi(z) = a_0 \log(z) + \sum_{l=1}^{\infty} \frac{b_l}{z^l}
\]

\[
b_l = -\frac{a_0 z_0^l}{l} + \sum_{k=1}^{l} a_k z_0^{l-k} \binom{l-1}{k-1} \quad \text{with} \quad \binom{l-1}{k-1} \text{ the binomial coefficients}
\]
### FSI Algorithm details

#### Fluid Flow

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega + \frac{1}{\rho} \nabla \times \nabla p
\]

#### Fluid to Solid

\[
\frac{d(M_x)}{dt} = F
\]

\[
\frac{d(I\omega)}{dt} = \tau
\]

#### Solid to Fluid

\[
\mathbf{u} = \mathbf{u}_S \equiv \mathbf{u}^f + \mathbf{u}^{\text{def}}
\]

#### Single-fluid model for FSI

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega
\]

\[
- \nabla \rho \times \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} - \mathbf{g} \right) + \chi (\mathbf{u}_S - \mathbf{u})
\]

\[
\mathbf{u}_i^r = \frac{1}{m_i} \int_{\Omega} \rho \chi_i \mathbf{u} \, dx
\]

\[
\dot{\theta}_i = \int_{\Omega} \rho \chi_i (\mathbf{x} - \mathbf{x}_i) \times \mathbf{u} \, dx
\]

\[
\mathbf{u}_i^r = \dot{\theta}_i \times (\mathbf{x} - \mathbf{x}_i)
\]

#### Discretization

|\[ t^n \] | Refinement | $\mathcal{R}(\omega^n$ and/or $\mathbf{u}^n)|_{t^n}$ |
|-----------|-------------|----------------------------------|
| Velocity  | $\nabla \psi^n = -\omega^n$ | $\mathbf{u}^n = \nabla \times \psi^n$ |
| Penalty   | $\rho^n = \rho_f + \sum_{i=1}^{N} (\rho_i^n - \rho_f) \chi_i^n$ |
| Projection| $\mathbf{u}_i^{n,1} = \frac{1}{m_i} \int_{\Omega} \rho^n \chi_i^n \mathbf{u}^n \, dx$ |
| Stretching| $\frac{\partial \omega^n}{\partial t} = \nabla \cdot \nabla \mathbf{u}^n$ |
| Baroclinity| $\mathbf{u}_\lambda = \frac{\mathbf{u}^n + \lambda \delta t \mathbf{u}_S}{1 + \lambda \delta t}$ | $\omega_\lambda = \omega^n + \nabla \times (\mathbf{u}_\lambda - \mathbf{u}^n)$ |
| Diffusion | $\frac{\partial \omega_\lambda^n}{\partial t} = \nu \nabla^2 \omega_\lambda^n$ |
| Advection | $\frac{\partial \omega_\lambda^n}{\partial t} + \mathbf{u}_\lambda^n \cdot \nabla \omega_\lambda^n = 0$ | $\omega^{n+1}_\lambda = \omega^{n+1}$ |
| $x_{i}^{n+1} = x_{i}^{n} + \mathbf{u}_i^n \Delta t^n$ | $\theta_i^{n+1} = \theta_i^n + \dot{\theta}_i^n \Delta t^n$ |
| Compression | $\mathcal{C}(\omega^{n+1}$ and/or $\mathbf{u}^{n+1})|_{t^n}$ |

<table>
<thead>
<tr>
<th>[ t^{n+1} ]</th>
<th>Explicit Euler</th>
<th>Explicit Euler</th>
<th>Explicit Euler</th>
<th>Explicit Euler</th>
<th>Explicit Euler</th>
</tr>
</thead>
</table>

- **Multipoles**: $\mathcal{R}(\omega^n$ and/or $\mathbf{u}^n)|_{t^n}$
- **RK2**: $\mathbf{u}_\lambda = \frac{\mathbf{u}^n + \lambda \delta t \mathbf{u}_S}{1 + \lambda \delta t}$
- **RK2+LTS**
- **RK2+LTS / particles**
Fish surface pressure-force

- Time varying force
  - Sum : Net $F_x$ and $F_y$
  - For validation against $a_{x/y}$ from velocity

- Thrust magnitude
  - Almost all segments contribute to thrust
  - Either side of head - thrust (negative pressure)
  - Largest drag: tip of head, and tail-end

![Pressure-force vector at $T = 0.010$](image1)

$\sum : \text{Net } F_x$ and $F_y$

![Localized thrust, $T = 0.010$](image2)
Pressure and thrust

• Thrust contribution: depends on
  • Pressure sign
  • Body curvature

• Regions close to head contribute to thrust via pressure
  • Caveat: still need to examine viscous contribution